

## A STUDY ON VECTOR FUNCTION SPACES

Varsha Agrawal, Research Scholar<sup>1</sup>

Dr. Alok Kumar, Professor<sup>2</sup>

Swami Vivekanand University, Sagar, M.P.<sup>1,2</sup>

### ABSTRACT

*This paper has been devoted to the study of vector function spaces on  $X$ . We define the Bishop and Silov decompositions in several ways for vector function spaces. If  $A$  denotes a complex function space on  $X$  then the tensor product  $A \otimes B$  of  $A$  and Banach algebra  $B$  can be regarded as a vector function space on  $X$ . The concept of the slice product  $A \widehat{\otimes} B$  of a function algebra  $A$  with a Banach algebra  $B$  has been defined earlier [1]. We extend the idea of  $A \# B$  for a complex function space  $A$  on  $X$ . Then  $A \# B$  also can be considered as a vector function space on  $X$  and  $A \widehat{\otimes} B \subset A \# B$ . Mainly, we concentrate our study to the decompositions and their properties for vector function spaces of the types  $A \widehat{\otimes} B$  and  $A \# B$ , where  $A$  denotes a complex function space on  $X$ . In particular, we show that the Bishop (Silov) decompositions for a complex function space  $A$  and the vector function space  $A \widehat{\otimes} B$  are the same.*

**KEYWORDS:** function spaces, function algebra, decompositions, Bishop and Silov decompositions

### 1.1 INTRODUCTION

Let  $X$  be a compact Hausdorff space and  $A$  be a linear subspace of  $C(X)$  that contains the constant functions.  $A_{\mathbb{R}}$  denotes the real functions in  $A$ , i. e.,  $A_{\mathbb{R}} = A \cap C_{\mathbb{R}}(X)$ .

The real [2] and complex [3] multipliers for  $A$  are defined respectively as,

$$M(A) = \{ f \in C_{\mathbb{R}}(X) : fg \in A \text{ for every } g \in A \} \text{ and}$$

$$N(A) = \{ f \in C(X) : fg \in A \text{ for every } g \in A \}.$$

**Definitions** [4]. (i) A subset  $K$  of  $X$  is said to be a set of antisymmetry in (FP)- sense or an (FP)-antisymmetric set for  $A$  if whenever  $f$  is in  $MCA|_K$ , then  $f$  is constant.

The collection of all maximal sets of antisymmetry in (FP)-sense forms a decomposition of  $X$ . We shall call this decomposition the Bishop decomposition in (FP)-sense for  $A$  and denote it by  $K_{FP}(A)$ .

(ii) A set of constancy of  $M(A)$  is called an (FP) - Silov set for  $A$ .

The collection of all maximal (FP)-Silov sets forms a decomposition of  $X$ , called the Silov decomposition in (FP)-sense for  $A$ . We shall denote it by  $F_{FP}(A)$ .

**Definition** [5]. A subset  $K$  of  $X$  is said to be a set of antisymmetry in (E)-sense or an (E)-antisymmetric set for  $A$  if  $f \in N(A)$  and  $f|_K$  is real-valued, then  $f$  is constant on  $K$ .

The collection of all maximal sets of antisymmetry in (E)-sense forms a decomposition of  $X$ . We shall call this decomposition the Bishop decomposition in (E)-sense for  $A$  and denote it by  $K_E(A)$ .

In a similar fashion we can define the Silov decomposition with the help of  $N(A)$ .

**Definition** A set of constancy of  $N(A)_{\mathbb{R}}$  is called an (E)-Silov set for  $A$ .

The collection of all maximal (E)-Silov sets forms a decomposition of  $X$  which we shall call the Silov decomposition in (E)-sense for  $A$  and denote it by  $F_E(A)$ .

Now onwards, we shall take  $A$  to be a function space on  $X$ , i.e.,  $A$  is a closed subspace of  $C(X)$  with  $1 \in A$ .

### Remarks

- (i)  $M(A)$  and  $N(A)$  are closed subalgebras of  $A_{\mathbb{R}}$  and  $A$  respectively, containing constants.
- (ii)  $M(A) = N(A)_{\mathbb{R}}$  and hence  $F_{FP}(A) = F_E(A)$  which we shall denote by  $F(A)$ . Since, for any subset  $K$  of  $X$ ,  $(NCA|_K)_{\mathbb{R}} \subset MCA|_K$ ,  $k_{FP}(A) \subset k_E(A)$ . Also, since  $N(A)$  is an algebra,  $k_E$

- (A)  $< (A) < ($  In fact,  $k_E(A)$  and  $k_E(A)$  are the usual Bishop and Silov decompositions for  $N(A)$ ).
- (iii) Since  $N(A)$  is a closed subalgebra of  $C_R(X)$  and  $k(A)$  is a decomposition consisting of sets of constancy of  $M(A)$ ,  $k(A)$  is an u.s.c. decomposition.
- (iv) If  $A$  is an algebra, then  $M(A) = A$  and for any subset  $K$  of  $X$ ,  $M(A)|_K = (N|_K)_R = (NA|_K)_R$ . Therefore,  $k_E(A) = k_{FP}(A) = 9fCA$ , where  $9CCA$  is the usual Bishop decomposition for  $A$ . Also,  $F(A)$  will be the usual Silov decomposition for  $A$ .
- (v) If  $A$  is a subspace of  $C_R(X)$ , then  $k_E(A) = f_E(A)$ .
- (vi) It has been shown by Feyel and Pradelle that  $k_{FP}(A)$  satisfies the (CD)-property for  $A$  [4]. However, looking to the proof of the result, it is clear that the authors have proved that  $k_{FP}(A)$  satisfies the stronger (GA)-property for  $A$  (that is, if  $\mu \in b(A)^\circ$  then  $\text{supp } \mu \subset K$  for some  $K \in \% CA$ );

Recently Yamaguchi and Wada have defined the idea of a set of antisymmetry for a closed subspace  $A$  of  $C(X)$  which separates the points of  $X$  [6]. Their definition is exactly the same as that of [4]. They have also proved that the corresponding decomposition has the (CGA)-property for  $A$  and its members are  $p$ -sets for  $A$ .

Paltineanu [7] has defined antialgebraic sets for a subspace of  $C_R(X)$ .

Let  $M$  be a closed subspace of  $C_R(X)$  with  $1 \in M$ . For a subset  $S$  of  $X$ , define

$G(S) = \{f \in M: \text{for each } g \in M, \text{ there exists } h \in M \text{ such that } fg = h \text{ on } S\}$ .

A subset  $S$  of  $X$  is called an antialgebraic set for  $M$  if  $f \in G(S)$  implies that  $f$  is constant on  $S$ .

The collection of all maximal antialgebraic sets for  $M$  is a decomposition of  $X$ .

Paltineanu [7] has shown that the above decomposition has the (GA)-property for  $M$ . However, it is easy to see that the Paltineanu's decomposition of  $X$  into maximal antialgebraic sets for a closed subspace of  $C_R(X)$  coincides with the Bishop decomposition in (FP)-sense. We have already noted that the latter has the (GA)-property. Thus Paltineanu's result is a special case of the result of Feyel and Pradelle [4].

We shall show that most of the results proved for a function algebra in section 1 of the first chapter are also valid for a subspace of  $C(X)$ .

If there is no danger of confusion regarding the subspace  $A$  under discussion, then we shall write  $k_{FP}, k_E$  and  $F$  instead of  $k_{FP}(A), k_E(A)$  and  $k(A)$  respectively.

**Theorem.** Let  $A$  be a function space on  $X$  and  $l$  be an u.s.c. decomposition of  $X$  with the (D)-property for  $A$ . Then  $k < l$ .

**Proof.** Let  $F \in k$ . For  $f \in C_R(X/l)$ , we have  $f \circ q \in C_R(X)$ , where  $q: X \rightarrow X/l$  is the quotient map. Also,  $(f \circ q)|_{g_S}$  is constant, say  $\alpha_S$ , for each  $S \in l$ . Hence  $f \circ q \in A$ , by the (D)-property. To show that  $f \circ q \in (N(A))_R$ , let  $h \in A$ . Then  $f \circ q \cdot h|_S = \alpha_S h|_S \in A|_S$  for all  $S \in l$ . Since  $l$  has the (CD)-property for  $A$ ,  $(f \circ q) \cdot h \in A$ . Hence  $f \circ q \in N(A)$  and so, as in the proof of Theorem,  $F \subset S$  for some  $S \in l$ .

The following corollary is immediate.

**Corollary** (i)  $k_{FP} = F$  if and only if  $k_{FP}$  is u.s.c. and (ii)  $k_E = F$  if and only if  $F$  is u.s.c..

Note that if  $k_{FP}$  is u.s.c., then  $k_{FP} = F$  and hence

$k_{FP} = k_E = F$ . Thus if  $k_{FP}$  is u.s.c., then  $k_E$  is u.s.c..

Next corollary shows that the Bishop decompositions determine the Silov decomposition. The proof is similar to the proof of Corollary. Let  $A$  and  $A_2$  be function spaces on  $X$ . Then

$k_{FP}(A_1) = k_{FP}(A_2) \rightarrow k(A_1) = F(A_2)$  and

$k_E(A_1) = k_E(A_2) \rightarrow k(A_1) = F(A_2)$ .

The converse of the above corollary is not true.

We shall give an example in which  $k_E(A_1) = k_E(A_2)$  but  $k_{FP}(A_1) \neq k_{FP}(A_2)$  and also give example in which  $k_{FP}(A_1) = k_{FP}(A_2)$  but  $k_E(A_1) \neq k_E(A_2)$ .

Note that, by Remark (i) and Theorem, if  $f$  is finite, then  $X - X$ . The following theorem shows that somewhat stronger result is true.

**Theorem** If  $y$  is finite, then  $k_{FP} = F$

**Proof.** Suppose that  $F = \{F_1, F_2, \dots, F_n\}$ . Now,  $M(A)$  is an algebra and hence, as in the proof of Theorem, for each  $i = 1, 2, \dots, n$ ; there exists  $f_i \in MCA$  such that  $f_i = 1$  on  $F_i$  and  $f_i = 0$  on  $F_j$  for every  $j \neq i$ .

Fix  $F_j$ . let  $g \in M(A|F_j)$ . Then there exists  $g \in A$  such that  $g|F_j = g$ . Since  $f_j \in M(A)$ ,  $g f_j \in A$ . To show that  $g'f_j \in M(A)$ , let  $h \in A$ . Then  $(g' f_j)h|F_k = [(g' h) | F_k] = 0$  if  $k \neq i$  and  $(g'f_j)h|F_i = (g'h)|F_i = g(h|F_i) \in A|F_i$ , as  $g \in M(A|F_i)$ . Since  $F$  has the (D) property for  $A$ ,  $(g f) h \in A$ . Hence  $g f_i \in MCA$ . Therefore,  $g f_i$  is constant on  $F_i$  and hence  $g$  is constant. This proves that  $F$  is a set of antisymmetry in (FP)-sense for  $A$ . Consequently,  $F_{FP}=F$

**Proposition** Let  $S$  be a CR set for  $A$ . Then

$$k_{FP}(A|_S) < k_{FP} \cap S, \quad k_E(A/) < k_E(A) \text{ and} \\ k/(A_S) < F(A) \cap S.$$

**Proposition** If  $S$  is a CR set for  $A$  which is saturated with  $k_{FP}(A)$  then  $k_{FP}(A|_S) = k_{FP}(A) \cap S$

## 1.2 GENERALIZATION OF A RESULT OF HAYASHI

In section 1, we have shown that the Silov decomposition is the finest u.s.c. decomposition with the (D)-property for a function space on  $X$ . Hayashi [8] has proved that the Bishop decomposition is the finest Hausdorff decomposable decomposition} with the (S)-property for a function algebra. We generalize the result for a function space on  $X$ . For the proof, we need certain properties of p-sets, which we discuss now.

Throughout this section,  $A$  denotes a function space on  $X$ . We begin with recalling some definitions.

A subset  $S$  of  $X$  is called a peak set for  $A$  if there exists  $f \in A$  such that  $f|_S = 1$  for all  $x \in X-S$ . The intersection of peak sets is called a generalized peak set for  $A$ . A closed subset  $S$  of  $X$  is called a p-set for  $A$  if  $A \in A^\perp$ , where  $\mu_S(G) = \mu(S \cap G)$  for every Borel set  $G$  of  $X$ .

### Remarks

- (i) Suppose  $S \subset T \subset X$  is a p-set for  $A$  and  $S$  is a p-set for  $A|_T$ , then  $S$  is a p-set for  $A$ .
- (ii) Let  $\{S_\alpha : \alpha \in \Lambda\}$  be a family of p-sets for  $A$ . Then  $S = \bigcap_{\alpha} S_\alpha$  is a p-set for  $A$  [8].

**Proposition** If  $S$  is a p-set for  $N(A)$ , then  $S$  is a P-set for  $A$ .

**Proof.** Let  $\mu \in T^\perp$  and  $\varepsilon > 0$ . Then there exists an open set  $U$  such that  $S \subset U$  and  $|\mu|(U-S) < \varepsilon$ . Since  $S$  is a p-set for  $N(A)$ ,  $S$  is an intersection of peak sets for  $N(A)$ . So, there exists a peak set  $T$  for  $N(A)$  such that  $S \subset T \subset U$  [9]. Let  $f \in N(A)$  be a peaking function for  $T$ . Then  $f^n$  converges to  $X_T$  boundedly and pointwise, where  $X_T$  is the characteristic function of  $T$ . Let  $g \in A$ . Then  $gf^n \in A$  and hence  $\int T g d\mu = \int X g d_T \mu = \lim \int X g f^n \mu = 0$ . But  $|\int S g f \mu - \int T g d \mu| < \varepsilon |g|$  [This implies that  $\int S g d\mu = 0$ ,  $\varepsilon$  as  $s$  is arbitrary. Since this is true for every  $g \in A$ ,  $\mu_S \in A^T$ .

Having discussed the p-sets for  $A$ , we go to the main result of this section. We start with a lemma.

**Lemma** If a decomposition  $\wedge$  of  $X$  has the (D)-property for  $A$ , then  $\wedge$  has the (D)-property for  $N(A)$

**Proof.** Let  $f \in C(X)$  and  $f|_S \in (NA)|$  for all  $S \in \wedge$ . Also, let  $g \in A$ . Since  $\wedge$  has the (D)-property for  $A$ , it suffices to show that  $fg|_S \in (A|_S)^\perp$  for all  $S \in \wedge$ . Let  $S \in \wedge$ . Since  $f|_S \in (N(A)|_S)^\perp$  there exists a sequence  $\{h_n\}$  in  $N(A)$  such that  $h_n$  converges uniformly to  $f$  on  $S$ . Then  $gh_n \in A$  and therefore,  $gh_n \in A|_S$  for each  $n$ . Since  $gh_n$  converges to  $gf$  uniformly on  $S$ ,  $fg|_S \in (A|_S)^\perp$ .

We now define a Hausdorff decomposable decomposition.

Let  $Y$  be a topological space. Two points  $y_1, y_2 \in Y$  are said to be H-equivalent if  $f(y_1) = f(y_2)$  for all  $f \in C_R(Y)$ , the set of all real-valued, bounded continuous functions on  $Y$ . The collection of all H-equivalence classes will be denoted by  $X(Y) = \{Y_\alpha\}$ , and we define the decomposition  $x_\alpha = \{Y_\alpha\}$  of  $Y$  for each ordinal number  $\alpha$ , by transfinite induction, as follows :

- (i) If  $\alpha = 0$ , then  $x_0 = \{Y\}$ ;
- (ii) If  $\alpha$  does not have the immediate predecessor, then  $x_\alpha = \bigcap_{\beta < \alpha} x_\beta$ , where  $\bigcap_{\beta < \alpha} x_\beta = \{ \bigcap_{\beta < \alpha} Y_\beta : Y_\beta \in x_\beta \}$ ;
- (iii) If  $\alpha$  has the immediate predecessor  $\beta$ , then  $x_\alpha = \bigcup_{Y_\beta \in x_\beta} x(Y_\beta)$ .

The process of defining a new partition, as above, from the previous ones will eventually stabilize, since the number  $\alpha$  is limited by the number of partitions of  $Y$ . Hence there exists a minimum ordinal number  $\alpha$  such that  $x_\alpha = x_{\alpha+1}$ . We denote this ordinal by  $\sigma(Y)$ . The decomposition

$x_{\sigma(Y)}$  is called the Hausdorff decomposition of  $Y$ . If  $x_{\sigma(Y)}$  consists of singleton sets  $\{y\}$  for all  $y \in Y$ , then  $Y$  is said to be

(a) Hausdorff decomposable space.

For a decomposition  $\delta$  of  $X$ , let  $Y = X/\delta$ . For each ordinal  $\alpha$ , define  $\widetilde{X}_\alpha = \{ \widetilde{Y}_\alpha : Y_\alpha \in X_\alpha \} = \{ q^{-1}(Y_\alpha) : Y_\alpha \in X_\alpha \}$  where  $q : X \rightarrow X/\delta$  is the quotient mapping and  $X_\alpha$  is the decomposition of  $Y$  as defined above.  $X_\alpha$  is called the decomposition of  $X$  associated with the decomposition  $X_\alpha$  of  $X/\delta$ . It is clear that  $\widetilde{x}_{\sigma(Y)} \delta$  if and only if  $X/\delta$  is a Hausdorff decomposable space.

We say that a decomposition  $\delta$  of  $X$  is Hausdorff decomposable if  $x_{\sigma(X/\delta)} = \delta$ .

It is clear that if  $X$  is a compact Hausdorff space and  $\delta$  is an u.s.c. decomposition of  $X$ , then  $\delta$  is Hausdorff decomposable.

Finally, we recall definition (ii) that a decomposition  $\delta$  of  $X$  has the (S)-property for  $A$  if for any  $p$ -set  $T$  which is saturated with  $\delta$ ,  $\delta \cap T$  has the (D)-property for  $(A/T)^-$ .

**Theorem.** Let  $A$  be a function space on  $X$  and  $\delta < k_{FP}$  be a decomposition of  $X$  with the (S)-property for  $A$ . Then,  $\widetilde{x}_{\sigma(Y)} = k_{FP}$ , where  $\widetilde{x}_{\sigma(Y)}$  is the decomposition of  $X$  associated with the decomposition  $\widetilde{x}_{\sigma(Y)}$  of  $Y = X/\delta$ .

**Proof.** Let  $q$  be the quotient map from  $X$  onto  $Y$  and for each ordinal  $\alpha$ , let  $\widetilde{x}_\alpha = \{ \widetilde{Y}_\alpha : Y_\alpha \in x_\alpha \}$  be the decomposition of  $X$  associated with the decomposition  $X_\alpha$  of  $Y$ . First we shall show that

- (i)  $\widetilde{Y}_\alpha$  is a  $p$ -set for  $A$ , for each  $Y_\alpha \in x_\alpha$  and it is saturated with  $\delta$ ;
- (ii)  $k_{FP} < \widetilde{x}_\alpha$ .

If  $\alpha = 0$ , then  $\widetilde{x}_\alpha = \{X\}$  and hence clearly (i) and (ii) hold. We assume that (i) and (ii) hold for all  $\beta < \alpha$ . If  $\alpha$  does not have an immediate predecessor, then by the definition,  $x_\alpha = \bigwedge_{\beta < \alpha} x_\beta$ .

Therefore,  $\widetilde{x}_\alpha = \bigwedge_{\beta < \alpha} \widetilde{x}_\beta$ , i.e.

$\widetilde{Y}_\alpha = \bigcap_{\beta < \alpha} \{ \widetilde{Y}_\alpha \in \widetilde{x}_\beta : \widetilde{Y}_\beta \subset \widetilde{Y}_\alpha \}$ . By Remark (ii),  $\widetilde{Y}_\alpha$  is a  $p$ -set and it can be seen that  $\widetilde{Y}_\alpha$  is saturated with  $\delta$ . Thus (i) holds. Since  $k_{FP} < \widetilde{x}_\beta$  for all  $\beta < \alpha$ ,  $k_{FP} < \bigwedge_{\beta < \alpha} x_\beta = \widetilde{x}_\alpha$ .

Suppose  $\alpha$  has the immediate predecessor  $\beta$ . Let  $K \in X_{FP}$ . Then  $K \subset \widetilde{Y}_\alpha$  for some  $\widetilde{Y}_\alpha \in k_{FP}$ . Then  $K \subset \widetilde{Y}_\beta$  for some  $\widetilde{Y}_\beta \in \widetilde{x}_\beta$ . Let  $f \in C_R(Y_b)$ . Then  $f \circ q \in C_R(\widetilde{Y}_\beta)$  and  $(f \circ q)|_E \in C_R(\widetilde{Y}_\beta)$  and  $(f \circ q)|_E$  is constant for every  $E \subset \widetilde{Y}_\beta$ ,  $E \in \delta$ . Since  $\delta$  has the (S)-property for  $A$ ,  $\delta \cap \widetilde{Y}_\beta$  has the (D)-property for  $A|_{\widetilde{Y}_\beta}$ . Hence, by Lemma,  $\delta \cap \widetilde{Y}_\beta$  has the (CD)-property for  $N(A|_{\widetilde{Y}_\beta})$ . Therefore,

$f \circ q \in N(A|_{\widetilde{Y}_\beta}) \cap C_R(\widetilde{Y}_\beta)$ , i.e.,  $f \circ q \in M(A|_{\widetilde{Y}_\beta})$  which implies that  $(f \circ q)|_K \in M(A|_K)$ . Hence  $(f \circ q)|_K$  is constant. This holds for any  $f \in C_R(Y_b)$ . Thus  $q(K) \subset Y_b$  for some  $Y_\alpha \in X_\alpha$ . So, we have  $K \subset \widetilde{Y}_\alpha \subset \widetilde{Y}_\alpha$ , and hence (ii) holds. Also, it can be seen that  $\widetilde{Y}_\alpha$  ( $\widetilde{Y}_\alpha \in X_\alpha$ ), is saturated with  $\delta$ . Thus to prove (i), by Remark (i), it suffices to show that  $\widetilde{Y}_\alpha$  is a  $p$ -set for  $A|_{\widetilde{Y}_\alpha}$ . In view of Proposition, it is enough to prove that  $\widetilde{Y}_\alpha$  is a  $p$ -set for  $N(A|_{\widetilde{Y}_\alpha})$ . Let  $Y_b/H$  be the quotient space of  $Y_b$  which is obtained by  $H$ -equivalence and  $p: Y_b \rightarrow Y_b/H$  be the natural quotient mapping. Since  $Y_b/H$  is a compact Hausdorff space, singleton set  $\{p(Y_\alpha)\}$  in  $Y_b/H$  is an intersection of peak sets  $\{S_a\}$  for  $C_R(Y_b/H)$ . But, if  $f \in C_R(Y_b/H)$ , then  $f \circ p \circ q \in C_R(\widetilde{Y}_\beta)$  and  $f \circ p \circ q|_E$  is constant for every  $E \subset \widetilde{Y}_\beta$ ,  $E \in \delta$ . Therefore, by the same argument as above,  $f \circ p \circ q \in C_R(Y_b/H) \circ p \circ q \subset N(A|_{\widetilde{Y}_\beta})$  and so,  $\{ (p \circ q)^{-1}(S_a) \}$  are peak sets for  $N(A|_{\widetilde{Y}_\beta})$ . Since  $\widetilde{Y}_\beta = (p \circ q)^{-1}(p(Y_\alpha))$  is the intersection of  $\{ (p \circ q)^{-1}(S_a) \}$ ,  $\widetilde{Y}_\beta$  is the intersection of peak sets or a  $p$ -set for  $N(A|_{\widetilde{Y}_\beta})$  which proves (i).

Now,  $X_{FP} < \widetilde{x}_{\sigma(Y)}$  If is strictly finer than  $\widetilde{x}_{\sigma(Y)}$  then there exists  $\widetilde{Y}_T \in \widetilde{x}_{\sigma(Y)}$  such that  $\widetilde{Y}_T = U_\alpha K_\alpha$  for certain  $K_\alpha \in k_{FP}$ , i.e.,  $\widetilde{Y}_T$  is not an (FP)-antisymmetric set for  $A$ . Hence there exists  $f \in M(A|_{K_\alpha})$  such that  $f$  is not constant. But  $f|_{K_\alpha} \in M(A|_{K_\alpha})$  and so,  $f|_{K_\alpha}$  is constant for each  $K_\alpha \subset \widetilde{Y}_T$ . Therefore,  $f$  defines a nonconstant real function on  $Y_T \in \widetilde{x}_{\sigma(Y)}$ . This contradicts the definition of  $\widetilde{x}_{\sigma(Y)}$  and completes the proof.

Since  $X_{FP}$  has the (S)-property for  $A$ , by taking  $\delta = k_{FP}$  in the above theorem, it follows that  $k_{FP}$  is Hausdorff decomposable.

**Corollary.** Let  $A$  be a function space on  $X$  and  $\delta$  be a decomposition of  $X$  which has the (S)-property for  $A$  and  $\delta < k_{FP}$ . Then  $\delta = k_{FP}$  if and only if  $\delta$  is Hausdorff decomposable. Thus  $k_{FP}$  is the finest Hausdorff decomposable decomposition with the (S)-property for  $A$ .



**Proof.** As noted at the end of the last theorem, if  $k_{FP} = \delta$ , then  $\delta$  is Hausdorff decomposable. Conversely, suppose that  $\delta$  is Hausdorff decomposable. Then  $\tilde{x}_{\sigma(Y)} = \delta$ , where  $Y = X/\delta$ . But, by Theorem,  $\tilde{x}_{\sigma(Y)} = k_{FP}$  and hence  $k_{FP} = \delta$ .

If  $X$  is a metrizable space, then the proof of Theorem can be simplified with the help of the following proposition.

**Proposition.** Suppose that  $X$  is a metrizable space. Let  $\delta$  be a decomposition of  $X$  with the (GA)-property for  $A$ . Then each  $E \in \delta$  is a  $p$ -set for  $A$ .

**Proof.** We imitate the method of the proof of Theorem in [10]. Let  $E \in \delta$ . It is enough to show that  $\mu_E \in A^\perp$  whenever  $\mu \in b(A^\perp)$ . Let  $\mu = b(A^\perp)^e$ . Then there exists  $F \in \delta$  such that  $\text{supp } \mu \subset F$ . We have  $E = F$  or  $E \cap F = \emptyset$ . Hence  $\mu_E = \mu$  or  $\mu_E = 0$ . Hence  $\mu_E \in A^\perp$ . Let  $\mu \in b(A^\perp)$ . Since  $X$  is metrizable  $C(X)$  is separable and hence the weak\* topology on  $M(X)$  is metrizable. Since  $b(A^\perp)$  is the weak\* closed convex hull of  $b(A^\perp)^e$  in  $M(X)$ , by Choquet's theorem [11], there exists a regular Borel measure  $\lambda$  supported on  $(A^\perp)^e$  such that  $f(\mu) = \int_X f d\mu = \int_{b(A^\perp)^e} f(\vartheta) d\lambda(\vartheta)$  for every  $f \in C(X)$ . Now,  $E$  is a  $G_\delta$ -set and hence  $\int_E f d\mu = \int_X f x_E d\mu = \int_{b(A^\perp)^e} (f x_E)(\vartheta) d\lambda(\vartheta)$  for every  $f \in C(X)$ . But for  $\vartheta \in b(A^\perp)^e$ ,  $\vartheta_E = \vartheta$  or  $\vartheta_E = 0$ . Hence, for  $f \in A$ ,  $0 = \int_E f d\mu = \int_X f d\mu = \int_X (f x_E) d\lambda = \int_{b(A^\perp)^e} (f x_E)(\vartheta) d\lambda(\vartheta)$ . It follows that  $\int_E f d\mu = 0$  for each  $f \in A$ , i.e.,  $\mu_E \in A^\perp$ . 85

To see how the above proposition simplifies the proof of Theorem, we observe that to prove (ii) for  $\tilde{x}_\beta$  we need (i) and (ii) for  $\tilde{x}_\beta$ ,  $\beta < \alpha$ . Now, if we assume (ii) for  $\tilde{x}_\beta$ , by taking  $\delta = \tilde{x}_\beta$ , in the above proposition, (i) immediately follows for  $\tilde{x}_\beta$ . Hence (ii) can be established for  $\tilde{x}_\beta$ . Having proved (ii) for  $\tilde{x}_\beta$ , (i) for  $\tilde{x}_\alpha$ , also follows immediately from Proposition and no separate proof for this is necessary.

**Remarks.** (i) If  $k_E = k_{FP}$ , then, since  $k_E = k(N(A))$ , the usual Bishop decomposition for  $N(A)$ ,  $k_{FP}$  has the (S)-property for  $N(A)$ . Conversely, if  $k_{FP}$  has the (S)-property for  $N(A)$ , then, by Corollary applied to  $N(A)$ ,  $k_{FP} = k_{FP}(N(A)) = k_E$ . as  $k_{FP} < k_E$  Hence  $k_{FP} = k_E$  if and only if  $k_{FP}$  has the (S)-property for  $N(A)$ . We shall give an example where  $k_{FP} < k_E$  example (b) which shows that does not satisfy the (S)-property for  $N(A)$ , in general.

(ii) Since  $f = \tilde{x}_1$  and  $k_{FP} = \tilde{x}_{\sigma(Y)}$  for  $Y = X/k_{FP}$ , it follows from the proof of Theorem that the members of  $f$  and  $k_{FP}$ , are  $p$ -sets for  $A$ . Since  $k_E = k(N(A))$ , by Proposition, members of  $k_E$  are  $p$ -sets for  $A$  also.

We give examples of decompositions which are strictly finer than  $k_{FP}$

**Examples.** (i) Let,  $X = [0,1] \times [0,1]$  and  $m$  be the Lebesgue measure on  $X$ . Let  $A = \{f \in C_R(X) : \iint f(x,y)u(x)dm(x,y) = 0 \text{ for all } u \in C_R[0,1]\}$  Then  $A$  is a closed subspace of  $C_R(X)$  and  $k_{FP}(A) = \{F_x : 0 \leq x \leq 1\}$ , where  $F_x = \{(x,y) : 0 \leq x \leq 1\}$  Example (d)]. Let  $\delta = \{F_x : 0 < x \leq 1\} \cup \{(0,y) : 0 \leq y \leq 1\}$ . Then  $\delta$  is a decomposition of  $X$  and  $\delta \geq k_{FP}(A)$  It can be checked that  $\delta$  is Hausdorff decomposable. Hence, by Theorem,  $\delta$  does not have the (S)-property for  $A$ .

(ii) Consider the function algebra  $B$  of Example (iii). Then  $B$  is an antisymmetric algebra, i.e.,  $k(B) = \{X\}$ . Let  $\delta$  be the decomposition consisting of maximal weakly analytic sets for  $B$ . Then we have seen that  $\delta \leq k(B)$  and  $\delta$  has the (GA)-property for  $B$ . Hence  $\delta$  has the (S)-property for  $B$ . Thus, by Corollary,  $\delta$  is not Hausdorff decomposable. Also, by Corollary, we can say that there is no function space  $A$  on  $X$  for which  $k_{FP}(A) = \delta$ .

We can define the essential set of a function space on  $X$  as we have defined it for a function algebra. Thus if  $I$  denotes the largest closed ideal of  $C(X)$  contained in  $A$ , then  $E(A) = \{x \in X : f(x) = 0 \text{ for all } f \in I\}$  is called the essential set of  $A$ .

The existence of such  $I$  can be shown as it is shown for function algebras.

**Remarks.** (i) Since  $I \subset N(A)$ ,  $E(A) = E(N(A))$ , where  $E(N(A))$  denotes the essential set of  $N(A)$ . Hence, by Proposition,  $E(A)$  is a  $p$ -set for  $A$ .

(ii) If  $f \in C(X)$  and  $f|_{E(A)} = 0$  then  $f \in N(A)$ .

(iii) If  $F$  is a closed set such that  $f \in C(X)$  and  $f|_F = 0 \Rightarrow f \in A$ , then  $E(A) \subset F$ .

The following proposition can be proved exactly as in the case of a function algebra [9; of this thesis, Proposition].

**Proposition.** Let  $A$  be a function space on  $X$  and  $E(A)$  denote the essential set of  $A$ . Let  $P_{FP}$

$P_E$  and  $P$  denote respectively the union of all singleton sets of  $(A)$  and  $F(A)$ . Then  $E(A) = \overline{X - P_{FP}} = \overline{X - P_E} = \overline{X - p}$ .

### 1.3 DECOMPOSITIONS FOR THE SPACE OF AFFINE FUNCTIONS

Let  $K$  be a compact convex subset of a locally convex topological vector space and  $OK$  denote the set of extreme points of  $K$ . Also, let  $A(K)$  denote the space of continuous, real-valued affine functions on  $K$ . Ellis [12] has discussed the Bishop and Silov decompositions for  $A(K)$ . Since a function in  $A(K)$  is determined by its value on  $OK$ , it is natural to concentrate on the space  $A(K)|_{\overline{\partial K}}$ , where  $\overline{\partial K}$  denotes the closure of  $OK$ . It is easy to see that  $A(K)|_{\overline{\partial K}}$  is a closed subspace of  $C_R(\overline{\partial K})$  which contains constants. As we have noted earlier, Paltineanu [2] has defined the decomposition of  $X$  into maximal antialgebraic sets for a subspace of  $C_R(X)$ . Also, Feyel and Pradelle [4] have defined the Bishop and Silov decompositions for a cone of  $C_R(X)$ . In this section, we compare the decompositions for  $A(K)|_{\overline{\partial K}}$  defined by these authors ([3], [4], [2]).

There is a natural association between a function algebra  $A$  and  $A(Z)$  for a suitable compact convex set  $Z$ . We study the relation between the decompositions for  $A$  and for  $A(Z)$  which we shall use to construct some examples in the next section.

**Definition.** The centre of  $A(K)$  is the set of all  $f \in A(K)$  such that for each  $g \in A(K)$ , there exists  $h \in A(K)$  with  $(fg)|_{OK} = h|_{OK}$ . It is denoted by  $C(A(K))$ .

It is clear that  $C(A(K))|_{\overline{\partial K}}$  forms a uniformly closed subalgebra of  $C_R(\overline{\partial K})$  and contains constants.

For  $E \subset OK, \overline{coE}$  denotes the closed convex hull of  $E$ . For the definition of a split face of  $K$  and the theory related to  $A(K)$ , we refer to Alfsen [13].

Ellis [12] has defined the Bishop and Silov decompositions for  $A(K)$  as follows.

**Definitions.** (i) A subset  $E$  of  $\partial K$  is said to be a set of antisymmetry if whenever  $f \in A(K)$  and  $f|_{\overline{coE}} \in C(\overline{coE})$  then  $f|_E$  is constant.

If  $\{S_\alpha : \alpha \in \Lambda\}$  denotes the collection of all maximal sets of antisymmetry, then  $S_\alpha = \partial E_\alpha$ , where  $E_\alpha$  is some closed split face of  $K$ , for each  $\alpha \in \Lambda$ . The family  $\{E_\alpha : \alpha \in \Lambda\}$  is called the Bishop decomposition for  $A(K)$  and is denoted by  $f(A(K))$ .

(ii) The sets of constancy of  $C(A(K))|_{\overline{\partial K}}$  are  $\{\partial F_\alpha : \alpha \in \Lambda\}$ , where  $F_\alpha$ 's are closed split faces of  $K$ . These faces  $F_\alpha$  are called the faces of constancy for  $C(A(K))$ . The family  $\{F_\alpha : \alpha \in \Lambda\}$  is called the Silov decomposition for  $A(K)$  and is denoted by  $f(A(K))$ .

We shall use  $\delta$  and  $f$  for  $\delta(A(K))$  and  $f(A(K))$ . The members of  $\delta$  and  $f$  are pairwise disjoint and  $K = \overline{co}(\cup_\alpha F_\alpha) = \overline{co}(\cup_\alpha E_\alpha)$ . In fact,  $\cup_\alpha \partial F_\alpha = \partial K$  and  $\cup_\alpha \partial E_\alpha = \partial K$ .

Ellis [12] has shown that  $f \cap \overline{\partial K}$  is a decomposition of  $\overline{\partial K}$  and it has the (D)-property for  $A(K)|_{\overline{\partial K}}$ . He has also shown that, in general,  $\delta \cap \overline{\partial K}$  does not cover  $\overline{\partial K}$  and it may not have the (D)-property. He has also discussed conditions under which  $\delta \cap \overline{\partial K}$  does have the (D)-property for  $A(K)|_{\overline{\partial K}}$ .

The following definition is due to Ellis [12].

**Definition.** A subset  $E$  of  $\overline{\partial K}$  is called a weak set of antisymmetry if whenever  $f \in A(K)$  and  $f$  satisfies the condition that for each  $g \in C(\overline{coE})$ , there exists  $h \in A(K)$  with  $fg|_E = h|_E$ , then  $f$  is constant on  $E$ .

The maximal weak set of antisymmetry is of the form  $W_\alpha = T_\alpha \cap \overline{\partial K}$ , where  $T_\alpha$  is a closed split face of  $K$ . The family  $w(A(K))$  of all maximal weak sets of antisymmetry for  $A(K)|_{\overline{\partial K}}$  forms a decomposition of  $\partial K$ . Let  $F(A(K)) = \{T_\alpha : T_\alpha \cap \partial K \in w(A(K))\}$ .

As usual, we write  $w$  and  $f$  for  $w(A(K))$  and  $f(A(K))$ .

**Remarks** [12]. (i)  $\delta \subset f$  and  $f \subset \delta$ .

(ii) If  $\partial K$  is closed, then  $\delta = T$ ,

(iii)  $w$  has the (D)-property for  $A(K)|_{\overline{\partial K}}$ .

Paltineanu [2] has shown that the decomposition of  $\overline{\partial K}$  into maximal antialgebraic sets for  $A(K)|_{\overline{\partial K}}$  coincides with the decomposition  $w$  of  $\overline{\partial K}$  into maximal weak sets of antisymmetry for  $A(K)|_{\overline{\partial K}}$  defined above. So, by Remark, it follows that  $w$  coincides with the Bishop decomposition in

(FP)-sense for  $A(K) |_{\partial K}$ , i.e.,  $w(A(K)) = k_{FP}(A(K) |_{\partial K})$ .

The following proposition shows that  $f \cap |_{\partial K}$  is the Silov decomposition for  $A(K) |_{\partial K}$  as considered in section 1.

**Proposition.**  $f \cap |_{\partial K} = f(A(K) |_{\partial K})$ .

**Proof.** It is enough to show that  $M(A(K) |_{\partial K}) = (A(K) |_{\partial K})$  where  $M(A(K) |_{\partial K}) = \{f \in C_R(|_{\partial K}): fh \in A(K) |_{\partial K}\}$ . Let  $f \in M(A(K) |_{\partial K})$ . Since  $A(K) |_{\partial K}$  contains,  $M(A(K) |_{\partial K}) \subset A(K) |_{\partial K}$  and so, there exists  $f \in A(K)$ . Then  $g|_{\partial K} \in A(K) |_{\partial K}$  and therefore,  $f(g|_{\partial K}) \in A(K) |_{\partial K}$ , i.e.,  $f'g' |_{\partial K} \in A(K) |_{\partial K}$ . Hence  $f' \in C(A(K))$  and  $f = f'g' |_{\partial K} \in C(A(K) |_{\partial K})$ . Thus  $M(A(K) |_{\partial K}) \subset C(A(K) |_{\partial K})$ .

Conversely, let  $f \in C(A(K))$  and  $f|_{\partial K} = f'$ . Then, clearly,  $f' \in C_R(|_{\partial K})$ . Let  $g' \in A(K) |_{\partial K}$ . Then there is  $g \in A(K)$  such that  $g|_{\partial K} = g'$ . But, then  $fg|_{\partial K} \in A(K) |_{\partial K}$ , as  $f \in C(A(K))$ . Therefore,  $fg|_{\partial K} \in A(K) |_{\partial K}$ , i.e.,  $f'g' \in C(A(K) |_{\partial K})$ . Hence  $f' = f|_{\partial K} \in M(A(K) |_{\partial K})$  and consequently,  $C(A(K) |_{\partial K}) = M(A(K) |_{\partial K})$ .

Thus the decomposition of  $\partial K$  into maximal weak sets of antisymmetry and the restriction of the Silov decomposition for  $A(K)$  to  $\partial K$  coincide with the Bishop and Silov decompositions for  $A(K) |_{\partial K}$  according to the Definitions.

It is natural to ask when the Bishop and Silov decompositions for  $A(K)$  coincide. Ellis [14] has answered this question in terms of weak centrality.

**Definitions.** (i) [15] If  $F$  is a closed split face of  $K$ , then  $F_{\perp} = \{f \in A(K) : f|_F = 0\}$  is called a near-lattice ideal.

(ii) For a compact convex set  $K$ ,  $A(K)$  is said to be weakly central if whenever  $I$  and  $J$  are maximal near-lattice ideals in  $A(K)$  such that  $I \cap C = J \cap C$ ,  $C$  being the centre of  $A(K)$ , then  $I = J$ .

**Theorem** [14]. If  $A(K)$  is weakly central, then the Bishop and Silov decompositions for  $A(K)$  are equal.

In view of the above theorem, it is of interest to characterize the weak centrality of  $A(K)$ . We give one such characterization in the following proposition.

**Proposition.**  $A(K)$  is weakly central if and only if  $A(K)$  satisfies the following property :

If  $G$  and  $H$  are minimal closed split faces

(\*) of  $K$  contained in the same face of constancy

for  $C$ , then  $G = H$ .

**Proof.** Assume that  $A(K)$  has the (\*)-property. Let  $I$  and  $J$  be maximal near-lattice ideals in  $A(K)$  with  $I \cap C = J \cap C$ , i.e.,  $G_{\perp} \cap C = H_{\perp} \cap C$  for closed split faces  $G$  and  $H$  in  $K$ . Now,  $G$  and  $H$  are compact convex sets and so, it can be checked that  $G \cap \partial K = \partial K$  and  $H \cap \partial K = \partial K$ . By Krein-Milman theorem,  $\partial G \neq \emptyset$  and  $\partial H \neq \emptyset$ . Thus  $G \cap \partial K \neq \emptyset$  and  $H \cap \partial K \neq \emptyset$ . Since  $K = \overline{co}(U_{\alpha} F_{\alpha})$ ,  $F_{\alpha} \in f$ ,  $\partial K \subset U_{\alpha} F_{\alpha}$ , So,  $G \cap F_{\alpha} \neq \emptyset$  and  $H \cap F_{\beta} \neq \emptyset$  (for some  $\alpha$  and  $\beta$ ). Now,  $G, H, F_{\alpha}, F_{\beta}$  are closed split faces of  $K$  implies that  $G \cap F_{\alpha}$  and  $H \cap F_{\beta}$  are also closed split faces of  $K$  [13]. Also, we have  $G \cap F_{\alpha} \subset G$  and  $H \cap F_{\beta} \subset H$ . Since  $G_{\perp} = I$  and  $H_{\perp} = J$  are maximal near-lattice ideals in  $A(K)$ ,  $G$  and  $H$  are minimal closed split faces of  $K$  [13]. Thus  $G \cap F_{\alpha} = G$  and  $H \cap F_{\beta} = H$ . Suppose  $\alpha \neq \beta$ . Then there exists  $f \in C$  and constants  $\lambda$  and  $\delta$  such that  $f|_{F_{\alpha}} = \lambda$  and  $f|_{F_{\beta}} = \delta$ ,  $\lambda \neq \delta$ . Since  $G \subset F_{\alpha}$ ,  $f|_G = \lambda$  and also  $f|_H = \delta$ , i.e.,  $f|_G \neq f|_H$ . But  $f \in I$  implies that  $f|_G = 0$ . So,  $f|_H = 0$ . Thus  $f \in H_{\perp} = J$ . Hence  $I \cap C = J \cap C \subset C$ . By the same argument, we can show that  $J \cap C \subset I \cap C$ . Therefore,  $J \cap C = I \cap C$ . Since  $A(K)$  is weakly central,  $I = J$  or  $G = H$ . Hence  $A(K)$  satisfies the (\*)-property.

Conversely, suppose that  $A(K)$  is weakly central and  $G$  and  $H$  are minimal closed split faces of  $K$  contained in the same face of constancy  $F_{\alpha}$  for  $C$ . Let  $G_{\perp} = I$  and  $H_{\perp} = J$ . Then  $I$  and  $J$  are maximal near-lattice ideals in  $A(K)$ . Let  $f \in I \cap C$ . Then  $f|_G$  is constant, say  $\lambda$ . Therefore,  $f|_G = \lambda = f|_H$ . But  $f \in I$  implies that  $f|_G = 0$ . So,  $f|_H = 0$ . Thus  $f \in H_{\perp} = J$ . Hence  $I \cap C = J \cap C \subset C$ . By the same argument, we can show that  $J \cap C \subset I \cap C$ . Therefore,  $J \cap C = I \cap C$ . Since  $A(K)$  is weakly central,  $I = J$  or  $G = H$ . Hence  $A(K)$  satisfies the (\*)-property.

Given a function algebra, a space of affine functions can be associated with it, as the following construction shows.

Let  $A$  be a function algebra on a compact Hausdorff space  $X$  and  $S$  denote the state space of  $A$ , i.e.,  $S = \{\phi \in A^* : \phi(1) = \|\phi\| = 1\}$ . Let  $Z = \overline{co}(S \cup (-S))$ , the weak\* closed convex hull of  $S \cup (-S)$ . Then  $Z$  is a weak\* compact convex subset of  $A^*$ . Asimow [5] has shown that  $\theta: A \rightarrow A(Z)$ ,

defined by  $(ef)(z) = \text{Ref}(z)$ , for  $z \in Z$ , is a topological isomorphism of  $A$  onto  $A(Z)$ .

The Bishop and Silov decompositions for  $A$  and  $A(Z)$  are related as follows.

**Theorem** [12]. Let  $A$  be a function algebra on  $X$  and  $Z = \overline{\text{co}}(\text{Su}(-iS))$ . Then  $K \in k(A)$  (respectively  $f(A)$ ) if and only if  $K = K \cap X$  for some  $k \in |\delta(A(Z))$  (respectively  $f(A(Z))$ )

Since the members of  $\delta(A(Z))$  (and  $f(A(Z))$ ) are disjoint, the above correspondence is one-to-one also. Hence the following corollary is immediate.

**Corollary**,  $K(A) = F(A)$  if and only if  $\delta(A(Z)) = (A(Z))$ .

Also, it is shown in [12] that  $\delta(A(Z)) = t(A(Z))$

## 1.4 TENSOR PRODUCT

We know that the tensor product of two function algebras is a function algebra. Also, we have seen that if  $A$  and  $B$  are function algebras on  $X$  and  $Y$  respectively, then  $K(A \widehat{\otimes} B) = K(A) \times K(B)$  and  $f(A \widehat{\otimes} B) = f(A) \times f(B)$ . The tensor product can also be defined for function spaces. So, the natural question is 'what about the corresponding Bishop and Silov decompositions?' We prove that  $K_{fp}(A \widehat{\otimes} B) = K_{fp}(A) \times K_{fp}(B)$ ,  $K_E(A \widehat{\otimes} B) = K_E(A) \times K_E(B)$  and  $f(A \widehat{\otimes} B) = f(A) \times f(B)$ , where  $A$  and  $B$  are function spaces on  $X$  and  $Y$  respectively. As usual,  $A \widehat{\otimes} B$  denotes the uniform closure of the space of all finite linear combinations of functions  $\{f \widehat{\otimes} g : f \in A, g \in B\}$ , where  $f \widehat{\otimes} g$  is a function on  $X \times Y$  defined by  $(f \widehat{\otimes} g)(x,y) = f(x)g(y)$ . It is easy to see that  $A \widehat{\otimes} B$  is a function space on  $X \times Y$ . For  $f \in A$  &  $B$ ,  $f_x \in B$  for all  $x \in X$  and  $f_y \in A$  for all  $y \in Y$ , where  $f_x(y) = f(x,y)$  ( $y \in Y$ ) and  $f_y(x) = f(x,y)$  ( $x \in X$ ).

**Lemma.** Let  $A$  and  $B$  be function spaces on  $X$  and  $Y$ . Then

(i)  $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$  and

(ii) if  $f \in N(A \widehat{\otimes} B)$ , then  $f_x \in N(B)$  for all  $x \in X$  and  $f_y \in N(A)$  for all  $y \in Y$

**Proof.** (i) Let  $f \in N(A)$  and  $g \in N(B)$ . Then, clearly,  $f \widehat{\otimes} g \in N(A \widehat{\otimes} B)$ . To show that  $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$ , let  $h \in N(A \widehat{\otimes} B)$ . Then  $h = \sum_{i=1}^n \phi_i \widehat{\otimes} \psi_i$  where  $\phi_i \in A$  and  $\psi_i \in B$  for all  $i$ . Thus  $(f \widehat{\otimes} g)h = (f \widehat{\otimes} g) \sum_{i=1}^n \phi_i \widehat{\otimes} \psi_i = \sum_{i=1}^n \phi_i f \widehat{\otimes} g \psi_i$ . Since  $f \in N(A)$  and  $g \in N(B)$ ,  $f \phi_i \in A$  and  $g \psi_i \in B$  for all  $i$ . Therefore,  $(f \widehat{\otimes} g)h \in A \widehat{\otimes} B$ . Now, if  $h \in N(A \widehat{\otimes} B)$ , then there is a sequence  $\{h_n\}$  in  $A \widehat{\otimes} B$  such that  $h_n \rightarrow h$  uniformly on  $X \times Y$ . But, as we have shown above,  $(f \widehat{\otimes} g)h_n \in A \widehat{\otimes} B$  for all  $n$  and hence  $(f \widehat{\otimes} g)h \in A \widehat{\otimes} B$ . Thus  $f \widehat{\otimes} g \in N(A \widehat{\otimes} B)$  and  $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$ . Since  $N(A \widehat{\otimes} B)$  is closed,  $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$ .

(ii) Let  $f \in N(A \widehat{\otimes} B)$  and  $g \in A$ . Fix  $y \in Y$ . Then  $(f_y g)(x) = (f_y g)(x) = f(x,y)(g \otimes 1)(x,y) = (f(g \otimes 1))y(x)$  for all  $x \in X$ . Since  $f \in N(A \widehat{\otimes} B)$ ,  $f(g \otimes 1)y \in A$ . Therefore,  $f_y g \in A$  and consequently,  $f_y \in N(A)$ . Similarly, we can prove that  $f_x \in N(B)$  for  $x \in X$ .

Next, we prove the result regarding the decompositions for the tensor product.

**Theorem.** Let  $A$  and  $B$  be function spaces on  $X$  and  $Y$  respectively. Then

(i)  $f(A \widehat{\otimes} B) = f(A) \times f(B)$ ;

(ii)  $k_E(A \widehat{\otimes} B) = k_E(A) \times k_E(B)$  and

(iii)  $k_{FP}(A \widehat{\otimes} B) = k_{FP}(A) \times k_{FP}(B)$

**Proof.** (i) By Lemma (i),  $(N(A) \widehat{\otimes} N(B))_R \subset (N(A \widehat{\otimes} B))_R$ . Therefore,  $f(A \widehat{\otimes} B) = f((N(A) \widehat{\otimes} N(B))_R) \subset f(N(A) \widehat{\otimes} N(B))$ , which is equal to  $f(A) \times f(B)$ , by Theorem. Hence  $f(A \widehat{\otimes} B) \subset f(A) \times f(B)$ .

Conversely, let  $G \in f(A)$  and  $H \in f(B)$ . It is enough to show that  $G \times H$  is a set of constancy of  $(N(A \widehat{\otimes} B))_R$ . Let  $f \in (N(A \widehat{\otimes} B))_R$ . Then, by Lemma (ii),  $f_x \in (N(B))_R$  and  $f_y \in (N(A))_R$  for each  $x \in X$  and  $y \in Y$ . So,  $f$  is constant on  $H$  for each  $x \in X$  and  $f_y$  is constant on  $G$  for each  $y \in Y$  which implies that  $f$  is constant on  $G \times H$ . Hence  $G \times H$  is a set of constancy of  $(N(A \widehat{\otimes} B))_R$ . Consequently, we get  $f(A) \times f(B) \subset f(A \widehat{\otimes} B)$ .

(ii) Since  $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$ , it is easy to verify that  $f(N(A \widehat{\otimes} B)) \subset f(N(A) \widehat{\otimes} N(B)) = f(N(A)) \times f(N(B))$ , (iii), where  $k$  indicates the usual Bishop decomposition. Thus  $k_E(A \widehat{\otimes} B) \subset k_E(A) \times k_E(B)$ .



Conversely, let  $G \in k(A)$  and  $H \in k_E(B)$ . To show that  $G \times H$  is an (E)-antisymmetric set for  $A \widehat{\otimes} B$ , let  $f \in N(A \widehat{\otimes} B)$  and  $f|_{G \times H}$  be real-valued. Then, by the same argument as in (i), we get  $f|_{G \times H}$  is constant. Hence  $k_E(A) \times k_E(B) \subset k_E(A \widehat{\otimes} B)$ .

Let  $G \in k_{FP}(A)$  and  $H \in k_{FP}(B)$ . Also, let  $f \in M(A \widehat{\otimes} B|_{G \times H})$ . But  $M(A \widehat{\otimes} B|_{G \times H}) \subset M((A/G)^- \widehat{\otimes} (B/H)^- \widehat{\otimes} N(A/G)^- \widehat{\otimes} (B/H)^-)_R$ . Therefore, (ii),  $f_x \in (N(A/G))^-_R = M(A/G)$  and  $f_x \in (N(B/H))^- = M(B/H)$  for each  $x \in G$  and  $y \in H$ . Hence  $f_x$  is constant on  $H$  and  $f_y$  is constant on  $G$  for each  $x \in G$  and  $y \in H$ . Thus  $f$  is constant. Therefore,  $G \times H$  is an (FP)-antisymmetric set for  $A \widehat{\otimes} B$ . Hence  $k_{FP}(A) \times k_{FP}(B) \subset k_{FP}(A \widehat{\otimes} B)$ .

Let  $T \in k_{FP}(A \widehat{\otimes} B)$ . It suffices to show that  $\pi_1(T)$  is an (FP)-anti symmetric set for  $A$ , where  $\pi_1: X \times Y \rightarrow X$  is the projection map. Let  $f \in M\{A|_{\pi_1(T)}\}$ . Then  $f \widehat{\otimes} 1 \in M((A \widehat{\otimes} B|_{\pi_1(T) \times Y}))$  and so,  $f \widehat{\otimes} 1|_T \in M(A \widehat{\otimes} B|_T)$ . Since  $T \in k_{FP}(A \widehat{\otimes} B)$ ,  $f \widehat{\otimes} 1|_T$  is constant and hence  $f$  is constant which shows that  $\pi_1(T)$  is an (FP)-antisymmetric set for  $A$ .

While we are discussing the tensor product of two subspaces, it would be of interest to know the relation between  $A(K_1) \widehat{\otimes} A(K_2)$  and  $A(CK_1 \times K_2)$ , where  $K_1$  and  $K_2$  are compact convex subsets of the same locally convex topological vector space and  $A(CK_1 \times K_2)$  is the set of all continuous, real-valued affine functions on  $K_1 \times K_2$ .

Let  $BA(K_1 \times K_2)$  denote the set of all biaffine functions on  $K_1 \times K_2$ , i.e., the set of all real-valued functions on  $K_1 \times K_2$  which are affine in each variable separately. Equivalently,

$$BA(K_1 \times K_2) = \{f \in C_R(K_1 \times K_2): f_x \in A(K_2) \text{ and } f_y \in A(K_1) \text{ for } x \in K_1, y \in K_2\}.$$

**Remarks.** (i) It is clear from the definition that  $A(K_1) \widehat{\otimes} A(K_2) \subset BA(K_1 \times K_2)$ .

(ii) It is easy to check that  $A(K_1 \times K_2) \subset BA(K_1 \times K_2)$ ,

However,  $A(K_1) \widehat{\otimes} A(K_2)$  may not be contained in  $A(K_1 \times K_2)$ .

**Example.** Let  $K_1=K_2 [0,1]$ . Define  $f: K_1 \rightarrow R$  by  $f(x) = x$  and  $g: K_2 \rightarrow R$  by  $g(y) = y$ . Then  $f \in A(K_1)$  and  $g \in A(K_2)$ . So,  $f \otimes g \in A(K_1) \widehat{\otimes} A(K_2)$ . But it is easy to see that  $f \otimes g \notin A(K_1 \times K_2)$ .

We do not know whether  $A(K_1 \times K_2) \subset A(K_2) \widehat{\otimes} A(K_1)$ .

We are now almost ready to give examples of function spaces for which at least two of the three decompositions  $k_{FP}$ ,  $k_E$  and  $f$  are not equal. We need one more lemma for that purpose.

**Lemma.** Let  $B$  be a closed subspace of  $C_R(X)$  with  $1 \in B$  and  $A = B + iB = \{f + ig: f, g \in B\}$ . Then  $f(A) = f(B)$ ,  $k_E(A) = k_E(B)$  and  $k_{FP}(A) = k_{FP}(B)$ .

**Proof.** First we show that  $N(A)$

$$= N(B) = iN(B) \dots\dots$$

Let  $f \in N(B) \subset B$  and let  $g \in A$ , i.e.,  $g = g_1 + ig_2$ , where  $g_1, g_2 \in B$ . Then  $fg_1$  and  $fg_2$  are in  $B$  and so,  $fg = fg_1 + ifg_2 \in B + iB = A$ . Therefore,  $f \in N(A)$  and  $N(B) \subset N(A)$ . Since  $N(A)$  is a complex subspace of  $C(X)$  containing constants,  $N(B) + iN(B) \subset N(A)$ . Conversely, let  $f \in N(A) \subset A$ . Therefore,  $f = f_1 + if_2$  with  $f_1, f_2 \in B$ . To show that  $f_1, f_2 \in N(B)$ , let  $g \in B$ . Then  $fg \in A$ , as  $B \subset A$  and  $f \in N(A)$ . Thus  $f_1g + if_2g \in B + iB$ . Since  $B \subset C_R(X)$  and  $f_1, f_2, g$  are in  $B$ ,  $f_1g$  and  $f_2g$  are in  $B$ . Hence  $f_1, f_2 \in N(B)$ . Thus (1) holds.

From (1) it is clear that  $(N(A))_R = N(B) = (N(B))_R$ . So,  $f(A) = f(B)$ . Also, for any closed subset  $G$  of  $X$ , we get  $N(A)|_G = N(B)|_G + iN(B)|_G$  and hence  $(N(A)|_G)_R = N(B)|_G = (N(B)|_G)_R$ . Thus  $f_E(A) = k_E(B)$ . Let  $G$  be a closed subset of  $X$ . Since  $A = B + iB$ ,  $A|_G = B|_G + iB|_G$ . So, as we have proved above,  $N(A|_G) = iN(A|_G) + iN(B|_G)$ . Therefore,  $M(A|_G) = (N(A|_G))_R = N(B|_G) = H(B|_G)$  and hence  $k_{FP}(A) = k_{FP}(B)$ .

The promised examples of subspaces now follow. We use the results of the previous section and technique of tensor product to construct the examples.

**Examples.** (a) Consider a function algebra  $A$  in which  $f(A) \neq f(A)$ , where  $k(A)$  is the usual Bishop decomposition for  $A$ . Then  $f_{FP}(A) = k_E(A) = f(A) \neq f(A)$ .

(b) Let  $A$  be a function algebra as in the above example. Let  $S$  be the state space of  $A$  and  $Z = \overline{c\partial}(S \cup (-iS))$ . Take  $B = A(Z)|_{\overline{\partial Z}}$ . In view of Corollary 5.3.10, we have  $\delta(A(Z)) \neq f(A(Z))$  and as

we have noted in section 3,  $\delta(A(Z)) k_{FP}(B)$ . Hence  $k_{FP}(B) \neq f(B)$ . Since  $B$  is a closed subspace of  $C_R(\partial Z)$ ,  $k_E(B) = k(B)$ . Hence  $k_{FP}(B) \neq k_E(B) = k(B)$ .

(c) Let  $A = B + iB$ , where  $B$  is as in example (b) above. Then, by Lemma,  $k_{FP}(A) \cong k_E(A)$ . Let  $C$  be a function space on  $Y$  such that  $k_E(C) \neq k(C)$  (Example (a) above). Then  $A \otimes C$  is a function space on  $X \times Y$  and in view of

**Theorem**,  $k_{FP}(A \otimes C) \cong k_E(A \otimes C) \cong (A \otimes C)$ .

The following examples show that the Bishop decomposition in (E)-sense neither determines nor is determined by the Bishop decomposition in (FP)-sense.

**Example.** Let  $A$  be a function space on  $X$  such that  $k_{FP}(A) \neq k_E(A)$ . Take  $B = N(A)$ . Then  $B$  is a closed subalgebra of  $C(X)$  and hence  $k_{FP}(B) = k_E(B)$ . But  $k_E(B) = k_E(A)$ . Therefore,  $k_{FP}(B)$ .

**Example.** As in Example, let  $X$  be the union of a line segment  $F$  and a sequence  $\{F_n\}$  of disjoint solid rectangles converging to  $F$ . Let  $A = P(X)$ , the usual polynomial uniform algebra on  $X$ . Let  $B$  be the set of all  $f$  in  $C(X)$  such that,  $f|_{F_n}$  is a polynomial of degree at most  $n$ , for each  $n$ . Then it can be checked that  $N(B) = \{f \in C(X) : f|_{F_n} \text{ is constant for each } n \in \mathbb{N}\}$ . Therefore,  $k_E(B) = \{F_n : n \in \mathbb{N}\} \cup \{F\}$ . But  $k_{FP}(B) = \{F_n : n \in \mathbb{N}\} \cup \{x : x \in F\} = k_{FP}(A) = k_E(A)$ . Thus  $k_{FP}(A) = k_{FP}(B)$  but  $k_E(B) \neq k_E(A) \neq k_E(B)$ .

We can define a set of antisymmetry in the usual way for function space  $A$  on  $X$ . That is, a subset  $K$  of  $X$  may be called a set of antisymmetry for  $A$  if whenever  $f \in A$  and  $f|_K$  is real, then  $f|_K$  is constant (as is defined for a function algebra). But then the corresponding decomposition may not have even the (D)-property for  $A$  (and so, it is not of much interest), as the following example shows.

**Example.** Let  $B$  be a proper closed subspace of  $C_R(X)$  which contains constants and separates the points of  $X$ . Take  $A = B + iB$ . Then  $A$  is a proper closed subspace of  $CCX$  with  $A_r = B$  and the decomposition consisting of maximal antisymmetric sets for  $A$  is  $\{x : x \in X\}$ . Since  $A \neq C(X)$ , this decomposition does not have the (D)-property for  $A$ .

**This example was suggested by referee [16].**

Let  $A$  be a function algebra on  $X$ . Then two well known subspaces  $(\text{Re}A)^-$  and  $A_R$  of  $C_R(X)$  are associated with  $A$ . It would be of interest to know the relation between the Bishop (Silov) decomposition for  $A$  and that of  $(\text{Re}A)^-$  and  $A_R$ . Note that, by Remarks (iv) and (v),  $k_{FP}(A_R) = k_E(A_R) = f(A_R)$  and  $k_E((\text{Re}A)^-)$ . To establish the relation, the following lemma is useful. The proof of this lemma is an imitation of Glicksberg's proof which is given for the decomposition  $k(A)$  [18].

**Lemma.** If a decomposition  $\delta$  of  $X$  has the (GA)-property for a function algebra  $A$ , then it has the (GA)-property for  $(\text{Re}A)^-$ .

**Proof.** Let  $\mu \in [b((\text{Re}A)^- \perp)]^c$  and  $S = \text{supp } \mu$ . Then clearly  $\mu \in b(A)$ . Suppose  $\mu = \lambda\eta + (1-\lambda)\eta$ , where  $v, \eta \in b(A^\perp)$  and  $0 < \lambda < 1$ . Then  $v, \eta \in M_R(X)$  and also,  $v, \eta \in ((\text{Re}A)^-)$ . That is,  $v$  and  $\eta$  are in  $b((\text{Re}A)^-)$ . Since  $\mu$  is an extreme point,  $(\mu - v - \eta)$ . Hence  $(\mu \in b(\text{Re}A))$ . Now,  $\delta$  has the (GA)-property for  $A$  implies that  $S \subset E$  for some  $E \in \delta$ . Thus  $\delta$  has the (GA)-property for  $(\text{Re}A)^-$ .

In view of the above lemma,  $k(A)$  and  $f(A)$  have the GA-property for  $(\text{Re}A)^-$  also. Hence  $k(A)$  and  $k(A)$  have the (S)-property and (D)-property for  $(\text{Re}A)^-$ . By Corollary 2.2.6,  $k_{FP}(\text{Re}A)^- < k(A)$  and by Theorem 2.1.7,  $f((\text{Re}A)^-) < f(A)$ .

In general,  $k_{FP}((\text{Re}A)^-) \cong k(A)$  and  $k(A)$  and  $f((\text{Re}A)^-) \cong f(A)$ .

**Example.** Let  $A = A(D)|_T$ , the disk algebra on the unit circle. Then  $A$  is a Dirichlet algebra [9]. Hence  $(\text{Re}A)^- = C_R(T)$ . So,  $k_{FP}((\text{Re}A)^-) = \{x : x \in T\}$

It is clear from the definitions that  $f(A) = f(A)_R$  and  $f(A)_R = k_E(A_R)$ . Therefore,  $k_E(A_R) = k_{FP}(A_R)$

**Remark.** In the above discussion, we have considered the case of a function algebra only. If  $A$  is a function space, then we should consider not only the subspaces  $A_r$  and  $(\text{Re}A)^-$  but also  $N(A_R)$ ,  $(N(A))_R$ ,  $N((\text{Re}A)^-)$  and  $(\text{Re}(N(A)))^-$ . As if this is not enough, we have also to reckon with possibly three different decompositions  $k_{FP}$ ,  $k_E$  and  $f$  for some of FP E these spaces. This would make the situation quite complicated and so, we have avoided its discussion.

## 1.5 CONCLUSIONS

In this paper, we study two Bishop type decompositions, defined by Feyel and Pradelle and by Edwards for a subspace  $A$  of  $COO$ . In section 1, we discuss the relation between these two types of Bishop decompositions and also their relation with the Silov decomposition. If  $A$  is a subspace of  $C_R(X)$ , then one of these Bishop decompositions coincides with the decomposition defined earlier by Paltineanu. The main theorem of section 2 states that the Bishop decomposition in (FP)-sense is the finest Hausdorff decomposable decomposition with the (S)-property for  $A$ . This generalizes a result of Hayashi. In section 3, we discuss the space  $A(fQ$  of continuous, real-valued, affine functions on a compact convex set  $K$ . In the last section, we prove that the Bishop and Silov decompositions of the tensor product of function spaces are the product of the corresponding decompositions. Finally, we use the results of section 3 and tensor product technique to give examples where at least one decomposition is different from the others.

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