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THE DEFICIENT DISCRET QUARTIC SPLINE INTERPOLATION OVER UNIFORM MESH FOR MODELS

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ABSTRACT

In this paper, we have obtained existence uniqueness and error bounds for discrete quartic spline interpolation for models system.

Keywords: - Quartic Spline Interpolation, Error Bounds and Models system

1. INTRODUCTION

Discrete spline has been introduced by Mangasarian and Shumaker [8] close connection with best summation formula which is a special case of abstract theory of best of approximation of linear functions. To compute nonlinear splines iteratively discrete cubic splines which interpolates given functional of a uniform mesh have been studied by Lyche [6]. These results were generalized by Dikshit and Powar [2] for non-uniform mesh. It has been observed that deficient spline more useful than usual spline as they require less continuity requirement at the mesh points. In the direction of some constrictive aspect of discrete splines Malcolm [7] has used discrete splines we refer to Astor and Duris [1], Jia [5] and Schumaker [12],Dubey and Nigam [3] and Dubey and Paroha [4].

2. EXISTENCE AND UNIQUENESS

Let a mesh on [01] be defined by P: $0 = x_0 < x_1 < \ldots\ldots\ldots\ldots.. x_n = 1$

Such that$x_i - x_{i-1} = P$ for $i = 0\ 1, 2, \ldots\ldots\ldots\ldots n-1$ (2.1)

Through h will represent given position real number. For a given function f, we have introduced the following interpolatory conditions

$$s(\alpha_i) = f(\alpha_i)\alpha_i = x_i + \frac{1}{3}\mathrm{P} \tag{2.2}$$

$$s(\beta_i) = f(\beta_i)\beta_i = x_i + \frac{1}{2}P \tag{2.3}$$

$$D_h^{\{1\}}s\{\gamma_i\} = D_h^{\{1\}}f\{\gamma_i\}\gamma_i = x_i + (\frac{1}{4})P \tag{2.4}$$

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and boundary conditions

$$s(x_0, h) = f(x_0, h) \qquad (2.5)$$

$$s(x_n, h) = f(x_n, h)$$

The class S(m, r, p, h) of deficient discrete spline of degree-m with deficiency r is the set of all continuous function s(x , h) such that for i = 0, 1, ………n-1 the restriction $s_i$of s(x, h) on $[x_i, x_{i+1}]$is a polynomial of degree m or less and

$$D_h^{\{i\}} s_i(x_i, h) = D_h^{\{i\}} s_{i+1}(x_i, h), \; j = 0, 1, \ldots\ldots m\text{-}r\text{-}1 \qquad (2.6)$$

Where the difference operator$D_h^{(i)}$ for a function is defined by

$$D_h^{\{0\}} f(x) = f(x) \text{ or } D_h^{\{1\}} f(x) = \frac{f(x+h)-f(x-h)}{2h}$$

We solve the following:

Problem 1: Given h > 0 for what restriction, there exist a uniques(x, h) ϵ S*(4, 1, P, h) which satisfies the condition (2.2) - (2.5).

Let P(t) be the quartic polynomial on [0, 1] then we can show that

$$P(t) = P\left(\frac{1}{3}\right)Q_1(t) + P\left(\frac{1}{2}\right)Q_2(t) + P'(1/4)Q_3(t) + P(0)Q_4(t) + P(1)Q_5(t) \qquad (2.7)$$

Where, $Q_1(t) = [\{(9/32)\}t + \{-9/32\}t^2 + \{-9/8\}t^3] + \{9/8\}z^4/A$

$$Q_2(t) = \left[(1/9)t + t^2\left\{28/9 - \frac{13}{3}\right\} + t^3\{31/9\} + t^4\left\{+ - \frac{71}{3}\right\}\right]/A$$

$$Q_3(t) = \left\{\frac{-t}{9} + \frac{2t^2}{3} - \frac{11t^3}{9} + \frac{2t^4}{3}\right\}/A$$

$$Q_4(t) = \left[1 + \{(-13/12)\}t + \{+41/18\}t^2 + \left\{-\frac{13}{12}\right\}t^3 + \{13/12\}t^4\right]/A$$

$$Q_5(z) = \left[\left\{-1/288 + \left(-\frac{1}{18}\right)h^2\right\}t + \left\{\frac{11}{288} - (11/18)h^2\right\}t^2 + \{-1/8 + 2h^2\}t^3 + \{1/8 - 2h^2\}t^4\right]/A$$

Where

$$A = \left[(5/144 + \left(-\frac{5}{9}\right)h^2\right]$$

Now we are set to answer the problem “1” in the following:

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Theorem 2.1: for any h>0 there exists an unique deficient discrete quartic splines(x, h) S(4, 1, P, h) which satisfies the condition (2.2) - (2.4).

Proof: Denoting $(x-x_i)/P$ by t, $0 \leq t \leq 1$ we can write (2.5) in the form of the restriction s_i (x, h) as he deficient discrete quartic spline s (x, h) on $[x_i,x_{i+1}]$ as follows.

$$s_i(x,h) = f(\alpha_i)Q_1(t) + f(\beta_i)Q_2(t) + P \quad f(\gamma_i)Q_3(t) + s_i(x_i)Q_4(t) + s_i(x_{i+1})Q_5(t) \tag{2.8}$$

From equation(2.6) we can easily verified that $s_i(x, h)$ is quartic on $[x_i, x_{i+1}]$ for i = 0, 1 … n-1 satisfying (2.2) - (2.5) we apply the continuity of first difference of $s_i(x, h)$ at x_i in (2.6) to see that

$$A_1(1/4,h)s_{i-1}+A_2(1/4,h)\, s_i \; +A_3(1/4,h)s_{i+1}=F_i,\ I =1,2…n \tag{2.9}$$

Where

$A_1(1/4,h) =$ [-7/24-h^2(22/9) +(271/36)h^2)]

$A_2(1/4,h)\ =$ [P^2 {103/96+(35/96) h^2 +{-569/72 +2h^2} h^2]

$A_3(1/4,h) =$ [P^2{3/72+h^2(1/18)} +{-17/72 +2h^2}h^2

and

$$\left(-\frac{449}{16}\right)P^2\{-(\frac{899}{8}+54h^2)f(\alpha_{i-1}) - \left\{9/32+\left(-\frac{9}{2}\right)h^2\right\}P^2 \ +\{9/8+18.h^2\}h^2f(\alpha_i)$$
$$+\left[\{(-151/15)+(128)h^2\}P^2 \ +\{-53/9+48h^2\}h^2\right]f(\beta_{i-1})$$
$$-f(\beta_i)\left[\{37/36+(16/9)h^2\}P^2 \ +(7/3-16h^2)h^2\right]$$
$$+f_{i-1}\left[\frac{2}{9}P^2 \ +\frac{13}{9}h^2\right]D_h^{\{1\}}f(\gamma_i) + \ D_h^{\{1\}}f(\gamma_i)P \quad \left[\frac{1}{9}P\text{^}2+\frac{11}{9}h^2\right]$$

We see that coefficient of matrix satisfied diagonally dominant properties i.e absolute value of coefficient dominant over the sum of the absolute values of coefficient matrix of system of equation (2.9) is diagonally dominantand hence invertible.

3. ERROR BOUNDS

It may be observed that system of equation (2.7) may be written as

$$A(h).\ M(h) = F \tag{3.1}$$

Where A(h) is coefficient matric and M(h) =s_i(x, h) However as already shown in proof of theorem 2.1 A(h) is invertible. Denoting the inverse of A(h) byA^{-1}(h) we note that row max

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norm || A^{-1}(h) || satisfies the following inequality.

Where,|| A^{-1} (h) || ≤ y (h) (3.2)

Where,Y(h) = max $\{C_i (h)\}^{-1}$

For convenience we assume in this section that 1=Mh, where M is positive integer. It is also assume that he mesh point $\{x_i\}$ are such that $x_i \epsilon$ [0, 1] fori=1, 2, ……..h where discrete interval [0, 1]h is the set of points {0, h, 2h,………Mh} for a function f and two district points x_1, x_2 in its domain the divided difference is defined by

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{(x_1 - x_2)}.$$

For convenience we write $f^{\{1\}}$ for $D_h^{\{1\}}$ f and w(f, p) is the modules of continuity of f. The discrete norm of a function f over interval [0, 1] is defined by

$$\|f\| = \frac{max}{x\epsilon[0,1]} |f(x)| \qquad (3.3)$$

Without assuming any smoothness condition on data f, we shall obtain in the following bounds of error function

e(x) = s(x, h) – f(x) over the discrete interval [0, 1]h

Theorem 3.1: Suppose s(x, h) is the deficient discrete quartic spline interpolation of theorem-2.1, then

$$\| e(x) \| \leq y(h)\, k\, (P, h)\, w\, (f^{\{1\}}, P) \qquad (3.4)$$

$$\| e(x) \| \leq k^* (P, h)\, w(f^{\{1\}}, P) \qquad (3.5)$$

and

$$\| e^1(x) \| \leq k^{**} (P, h)\, w(f^{\{1\}}, P) \qquad (3.6)$$

Wherek(P, h) k* (P, h) and k** (P, h) are some positive function of p and h

Proof: writing $f_i \{x_i\} = f_i$ we notice that the equation (3.1) may be written as

$$A(h).\, e(x) = f_i (h) – A(h) f_i = L_i \qquad (3.7)$$

Put e(x) = s(x, h) - f_i

We need the following result due to Lyche [6] to estimate R.H.S. of (3.7).

Lemma 3.1: Let $\{a_i\}_{i=1}^{m}$ and $\{b_j\}_{j=1}^{n}$ be given of non negative real numbers such that

$$\Sigma a_i = \Sigma b_j$$

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Then for any real valued function f defined on discrete interval [0, 1]h we have

$$|\sum_{i=1}^{m} a_i[x_{i,0}, x_{i,1}, \dots \dots .. x_{ik}]\, f - \sum_{j=1}^{n} bj\, [y_{j\,0}, y_{j\,1}, \dots \dots \dots . y_{jk}]\, f| \; < h\big[f^{(k)}, 1 - kh|\Sigma a_i|k\big| \tag{3.8}$$

Where x_{ik},$y_{jk}\epsilon$ [0, 1] for internal values of i, j and k.

It may be observed that R.H.S. of (4.3.7) is written as

$$(L_i) = \left|\sum_{i=1}^{5} a_i[x_{i0}, x_{i1}]f - \sum_{j=1}^{3} bj\big[y_{j\,0}, y_{j\,1}\big]f\right| \tag{3.9}$$

Where,

$$a_1 = P\ \ \{-117/576 - (73/36)h^2\}P^2 - h^2\{-189/72 + (-3)h^2\}$$

$$a_2 = P\ \ \left\{-\frac{29}{8} + (-\frac{22}{9})\right\}P^2 + h^2\left\{\frac{607}{36} - 12h^2\right\}$$

$$a_3 = P\ \ \left\{\frac{2}{9}P_{i-1}^2 + \frac{19}{9}h^2\right\}$$

$$a_4 = P\ \ \left[\{-1/54 + (-16/54)h^2\}P^2 + \left\{-37/54 + \frac{8}{3}h^2\right\}h^2\right]$$

$$a_5 = P\left[\{-1/432 - (-1/27)h^2\}P^2 + \left\{22/27 - \frac{14}{3}h^2\right\}h^2\right]$$

$$b_1 = P\ \ \left[\left\{77/48 + \left(-\frac{9}{4}\right)h^2\right\}P^2 + h^2\{9/34 + (-9)h^2\}\right]$$

$$b_2 = P^3\,[\{7/48 + (-3)h^2\}]$$

$$b_3 = P\ \ \left[\{-7/432 + (-7/3)h^2\}P^2 + \left\{79/108 + \left(\frac{4}{3}h^2\right)\right\}\right]$$

$$b_4 = P^3\left[-\frac{1}{9}\right]$$

$$b_5 = P^3\left(\frac{11}{9}\right)h^2$$

and

$x_{10} = P_{i-1}$, $x_{11} = x_i = x_{20} = x_{21}$

$x_{20} = x_{i-1}$, $x_{30} = \gamma_{i-1}$-h, $x_{31} = \gamma_{i-1}$+h

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$x_{40} = \alpha_i$, $x_{41} = \beta_i$ $x_{50} = \alpha_i$ $x_{51} = \alpha_{i-1}$

$y_{10} = \alpha_{i-1}$ $y_{11} = \beta_{i-1}$ $y_{20} = x_i$ $y_{21} = \alpha_i$

$y_{30} = x_i$ $y_{31} = \alpha_i$ $y_{40} = y_i - h$ $y_{41} = y_i + h$

$y_{50} = x_i - h$ $y_{51} = y_i + h$

$$\sum_{i=1}^{5} a_n = \sum_{j=1}^{5} b_j$$

$$= \Big[\{503/576 + (3/4)h^2\}P^2 + h^2\{9/16 + (-9)h^2\} + P_i\{-7/432 + (-7/3)h^2\}P^2 + \left\{-53/108 + \frac{4}{3}h^2\right\}h^2\Big]$$

$$= k\,(P,h) \qquad \text{(Say)} \tag{3.10}$$

Thus apply Lemna 3.1 in (3.10) for m=n=4 and k=1 to see that

$$|\, L_i \,| \le k\,(P, h)\; w\,(f^{\{1\}}|1-P\,|\,) \tag{3.11}$$

Now using the equation (3.2) and (3.11) in (3.7) we get

$$\|\, e(x_i) \,\| < y(h)\; k(P, h)\; w\,(f\{1\}, |\, 1-P\,|) \tag{3.12}$$

Thus in equality (3.4) of theorem (3.1) to obtain the bound of e(x) we replace $s_i(x, h)$ by $e(x_i)$ in equality (3.6) to get

$$e(x) = e(x_i)\, Q_4\,(t) = e(x_{i+1})\, Q_5\,(t) + M_i\,(f) \tag{3.13}$$

Where,

$M_i(f) = f(\alpha_i)\, Q_1\,(t) + f(\beta_i)\, Q_2\,(t) + P\, f^{\{1\}}\,(\gamma_i)\, Q_3\,(t) + f(x_{i-1})\, Q_4(t) + f(x_i)\, Q_5(t) - f(x)$

A Little conversation shows that $M_i(f)$ in (3.13) may be written in the form of divided difference as follows

$$|M_i(f)| = \textstyle\sum_{i=1}^{3} a_i [x_{i0}, x_{i1}] f - \sum_{j=1}^{2} b_j\, [y_{i0}, y_{i1}]\, f \tag{3.14}$$

Where,

$$a_1 = [t(-3/64 - (-3/4)h^2 + t^2\{3/64 - h^2(3/4)\} + t^3\{-99/32 - 3h^2\} + t^4\{-33/16 + 3h^2\}]$$

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$$a_2 = P \left[\left\{t(-1/12 + \left(-\frac{4}{3}\right)h^2\right\} + \{205/288 - (-19/18)h^2\}t^2 + \{-79/72 + 2h^2\}t^3 + \{-13/12 - 2h^2\}\mathrm{t}^4\right]$$

$$a_3 = P \left\{\frac{1}{9}t + \frac{2}{3}t^2 - \frac{11}{9}t^3 + \frac{2}{3}t^4\right\}$$

$$b_1 = P \left[t\left\{31/576 + \left(\frac{11}{36}\right)h^2\right\} + \{-11/576 + (19/36)h^2\}t^2 - \{-1/`16 + 4h^2\}t^3 - (32/172 - h^2)t^4\right]$$

$$b_2 = P \quad t\{-1/48 + (-5/9)h^2\}$$

And

$x_{10} = \alpha_i$ $x_{11} = \beta_i = x_{21}$

$x_{20} = x_{i-1}$; $x_{30} = \gamma_i - h$; $x_{31} = \gamma_{i+1}$

$y_{10} = \beta_i$; $y_{11} = x_i$; $y_{20} = x_{i-1}$; $y_{21} = x$

Clearly

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j$$

$$= \left[\{85/64 + (-7/12)h^2\}t + t^2\left\{-11/576 + (-\frac{7}{36})h^2\right\} + (1/16 - h^2)t^3 + ((1/12 + h^2)t^4\right]$$

= K* (P, h)

Therefore applying lemma (4.3.1) for m=3, n=2 and k=1 we get

$$| M_i(f) | < K^*(P, h)\ w\ (f^{\{1\}}, P) \qquad (3.15)$$

Finally applying bounds of (3.12) and (3.15) in (.3.13) we get inequality (3.5) w how (proceed to obtain an upper bound for $e^{\{1\}}(x)$ for this we use first difference operator in (2.6) and get

$$PD_h^{\{1\}} s_i(x, h) = f(\alpha_i)\ Q_1^{\{1\}}(t) + f\ (\beta_i)\ Q_2^{\{1\}}(t) + Pf^{\{1\}}(\gamma_i\ Q_3^{\{1\}}(t) + s_{i-1}(x)\ Q_4^{\{1\}}(t) + s_i(x)\ Q_5^{\{1\}}(t) \qquad (3.16)$$

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Replacing $s_i(x)$ by $e(x_i)$ we get

$$Pe^{\{1\}}(x) = e_{i-1}\, Q_4^{\{1\}}(t) + e_i\, Q_5^{\{1\}}(t) + U_i(f) \qquad (3.17)$$

Where,

$$U_i(f) = f(\alpha_i)\, Q_1^{\{1\}}(t) + f(\beta_i)\, Q_2^{\{1\}}(t) + P_f^{\{1}(\gamma_i)\, Q_3^{\{1\}}(t) + f_{i-1}\, Q_4^{\{1\}}(t) + f_i\, Q_5^{\{1\}}(t) - Pf^{\{1\}}(x)$$

Now rewriting $U_i(f)$ in terms of divided difference we have

$$|U_i(f)| = \left|\sum_{i=1}^{3} a_i\,[x_{i0}, x_{i1}]_f - \sum_{j=1}^{2} b_j\,[y_{j0}, y_{j1}]_f\right|$$

Where,

$$a_1 = P\ \ \{14/72 - (4/3)h^2\} + t\{(11/36) - (19/9)h^2\} + (3t^2 + h^2)\{83/24 + 2h^2\} + \{(-13/24) - 2h^2\}(4t)(t^2 + h^2)$$

$$a_2 = P\ \ \left[(\frac{5}{576}) + h^2(1/36) + \{(11/288) - (-11/18)h^2\}2t + \{-1/18 + h^2\}(t^2 + h^2) + \{-17/48 - h^2\}4t(t^2 + h^2)\right]$$

$$a_3 = P\ \ \left\{\frac{-1}{9} + \frac{4}{3}t - \frac{11}{9}(3t^2 + h^2) + \frac{8}{3}t(t^2 + h^2)\right\}$$

$$b_1 = P\ \ [3/64 + (-3/4)h^2 + t\{-3/32 + (-15/2)h^2\} + \{-3/16 + 4h^2\}(3t^2 + h^2) + (3/16 - 3h^2)\,4t(t^2 + h^2)]$$

$$b_2 = P\ \ [-1/48 + (-4/9)h^2]$$

It can be easilyseen that

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{2} b_j$$

$$= P\ \ [-49/16 + (-47/36)h^2 + t\{-3/32 + (3/2)h^2\} + \{-3/16 + 3h^2\} + \{3/16 - 3h^2\}\,4t(t^2 + h^2)]$$

and

$y_{10} = \alpha_i$ $\quad$ $y_{11} = \beta_i$

$y_{20} = x+h$ $\quad$ $y_2 = x-h$ $\quad$ $x_{10} = \beta_i x_{11} = x_i$

$x_{30} = \gamma_i$-h $\quad$ $x_{20} = \beta_i x_{21} = x_{i+1}$ $\quad$ $x_{30} = \gamma_i$-h, $x_{21 = x_{i+1}}$, $x_{31 = \gamma_i}$+h

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Using (3.15) and (3.17) in (3.16), we get inequality (3.6).

This completes proof of theorem 3.1.

4. Conclusion

We have obtained existence,uniqueness and error bound of discrete quartic spline interpolation which agrees at points 1/2 and 1/3 and first difference at ¼ and it will useful in various model system.

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