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A STUDY ON FUNCTION ALGEBRA

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ABSTRACT

In this paper aim of researcher is to study on real function algebras. It is possible to associate a complex function algebra with a given real function algebra by complexifying it. This technique of complexification is often employed to study the properties of a real function algebra. A good deal of work has been done in the field of real function algebras. Kulkarni and Srinivasan have defined the Bishop decomposition for real v function algebras. We introduce the Silov decomposition for real function algebras and prove some basic properties of it.

Keywords: function algebra, real function, Hausdorff space, silov decomposition

1.1 INTRODUCTION

Let X be a compact Hausdorff space and let C(X) ($C_R(X)$ denote the set of all complex-valued (real-valued) continuous functions on X. With usual operations of addition, multiplication andthe norm defined by

$$\|f\| = \sup\{|f(x)| : x \in X\}$$

for f ∈ C (X) ($C_R(X)$), C(X) ($C_R(X)$) is a complex (real) Banach algebra with identity. A function algebra on X is a closed subalgebra of C(X) which contains and separates the points of X.

A decomposition of X is a collection of disjoint closed subsets of X whose union is X. Subalgebras of C(X) ($C_R(X)$) and the decompositions of X are closely related. For example, a closed ideal of is determined by a closed subset of X. Now, if F is a closed subset of X, then we can associate with it the decomposition

$$\boldsymbol{\alpha} = |F| \cup \{|x|, x \in X - F\}$$

Thus every closed ideal is associated with a decomposition of X consisting of a closed set and singletonsoutside the closed set. If A is a closed subalgebra of $C_R(X)$ (aself-conjugate closed subalgebra of C(X) containing constants, then the sets of constancy of A gives a decomposition which is upper semicontinuous. Conversely, if α is an upper semicontinuous decomposition of X, then there exists a unique closed subalgebra of $C_R(X)$ containing constants whose sets of constancy are precisely the members of α [1]. This association of decompositions of X and subalgebras of $C_R(X)$ has been found very useful in the study of $C_R(X)$ as a direct sum of two subalgebras ([2], [3], [4], [5]).

The role of decompositions in the study of function algebras was highlighted by Silov [6] and more so by Bishop [7]. The Silov decomposition for a function algebra A on X consists of sets of constancy of $A_R = A \cap C_R(X)$. The Bishop decomposition for A consists of

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maximal sets of antisymmetry. Both these decompositions have the following crucial property:

If $f \in C(X)$ and $f|_E \in (A|_E)$ for every member E in the decomposition, then $f \in A$.

The above property is known as the (D)-property in the literature [8]. Once the importance of decompositions is recognised, it is natural to ask further questions. Some of the questions are :

- Are there decompositions, other than Silov and Bishop, associated with a function algebra which also have the (D)-property?
- Does a Bishop (Silov) decomposition have a stronger property than the (D)-property?
- How are Bishop, Silov and other decompositions related to each other? Do some of these decompositions determine the others?
- Does every member of a decomposition satisfying property such as (D)-property have any special property in relation to a function algebra? (For example, every member of Bishop decomposition of a function algebra is an intersection of peak sets).
- How are the decompositions of A and $\widehat{A}$ related, where $\widehat{A}$ 0is the algebra of Gelfand transforms of A ?
- Can the decompositions analogous to Silov and Bishop for a function algebra be defined for a function space? What are their properties?
- How about the decompositions for a real function algebra? for an algebra of vector-valued continuous functions ?

Throughout the paper, X denotes a compact Hausdorff space, A denotes a function algebra on X, i.e., a closed subalgebra of C(X) which contains constants and separates the points -of X and A_R denotes the algebra of real-valued functions in A.

1.2 SILOV DECOMPOSITION

The Bishop decomposition was introduced by Bishop in 1961 [7] and has been studied by various authors since then ([9], [8], [10], [11]). The Silov decomposition was introduced by Silov earlier [6], but appears to have received less attention in literature. We first prove some results for Silov decomposition analogous to those known for Bishop decomposition and then give conditions under which these two decompositions are equal. Note that, in general, these decompositions are not equal ([12], [13]).

We recall some definitions

1.2.1 DEFINITIONS

(i) A subset K of X is called a set of antisymmetry for A if $f \in A$ and $f|_K$ is real-valued implies that $f|_K$ is constant. The collection of' all maximal sets of antisymmetry for A forms a decomposition of X, called the Bishop decomposition for A. We denote this decomposition by K(A).

(ii) A set of constancy of A_R is called a Silov set for A. The collection of all maximal Silov sets for A is called the Silov decomposition for A and we denote it by F(A).

We shall write only K (respectively F) in place of K(A) (respectively F(A)) if it is clear from the context which function algebra A is being refered to.

Example A function algebra A is said to be antisymmetric if K(A) = (X). Hence for an antisymmetric function algebra, the Bishop and Silov decompositions coincide. But the following example shows that in general the Bishop and Silov decompositions are not equal.

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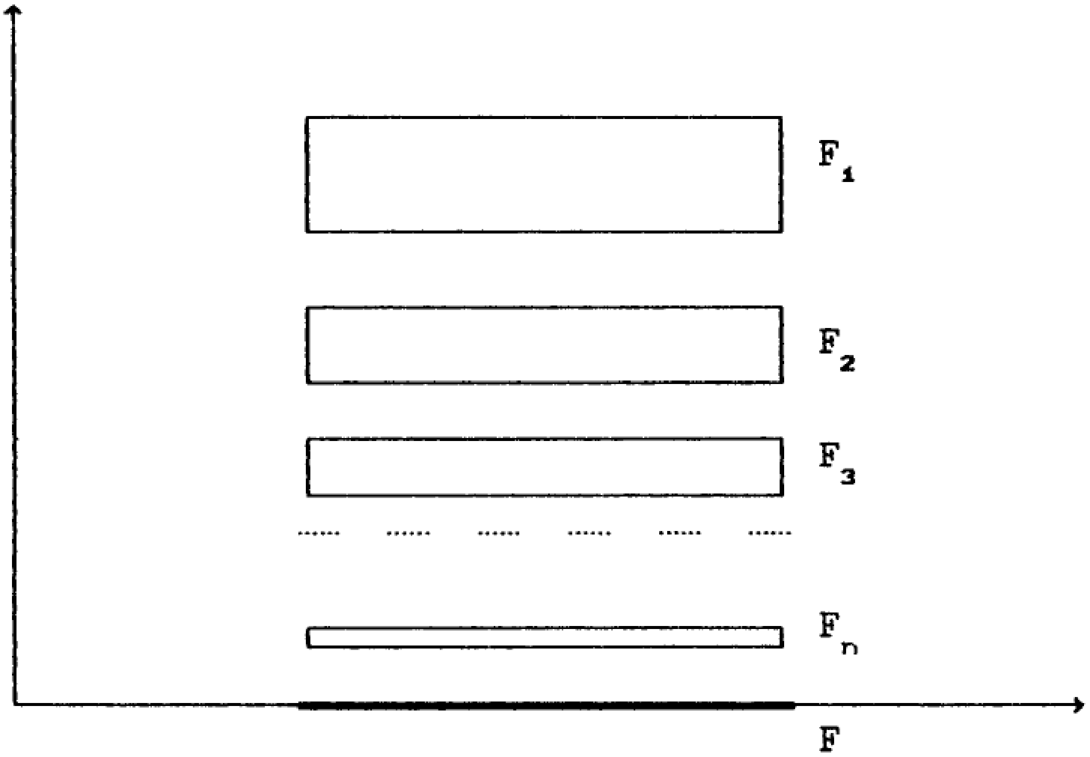

FIGURE 1.1

Let. X be the union of a line segment F and a sequence of disjoint solid rectangles $\{F_n$: n = 1,2, . . . } converging to F (see the Figure 1.1). Let A = P(X), the algebra of all functions in C(X) which can be uniformly approximated by polynomials in z. Then every real-valued function in A is constant on each F and hence on F. Therefore, the Silov decomposition F = $\{F_n : n = 1,2,......\} \cup \{\{x\} : x \in F\}$.

For a decomposition δ of X and a closed subset S of X, we have defined $\delta \cap S = \{E \cap S: E \in \delta$ and $\delta \cap S \neq \emptyset\}$. Then $\delta \cap S$ is a decomposition of S.

The following results about the Bishop decomposition can be found in [14].

1.2.2 THEOREM

Let A and B be function algebras on compact Hausdorff spaces X and Y respectively and % denote the Gelfand transform of A. Then

(i) Each K ∈ K is a p-set for A and hence K is a closed restriction set for A;

(ii) $K(\widehat{A}) = \{ \widetilde{K}: K \in K(A)$, where $\widetilde{K}$ is the A-hull of K and $K(A) = K(\widehat{A}) \cap X = \{\widetilde{K} \cap X: \widetilde{K} \in K(\widehat{A})\}$;

(iii) $K(A \widehat{\otimes} B = K(A) \times K(B)$.

Note that by definition, $K(\widehat{A}) \cap X = \{\widetilde{K} \cap X : \widetilde{K} \in K(\widehat{A}), \widetilde{K} \cap X \neq \emptyset$ and hence $K(\widehat{A}) \cap X = \{\widetilde{K} \in K(\widehat{A})\}$.

We show that similar results remain valid for the Silov decomposition also.

1.2.3 THEOREM

Let A be a function algebra on X. Then each F ∈ f is a p-set for A.

Proof Let F ∈ f and $f \in A_R$. Then $f|_F$ is constant, say α Also, let $F_f = \{x \in X: f(x) = \alpha \}$. Then F_f is a peak set for A with $g = 1 - \frac{(f-\alpha)^2}{\|(f-\alpha)^2\|}$ as a peaking function for F_f. Since this is true for every f $\in A_R$, it is enough to show that $F = \cap \{F_f : f \in A_R\}$ Clearly, $F \subset \cap \{F_f : F \in A_R \}$ Suppose $F \subsetneq \cap \{F_f : f \in A_R\}$. Then there exists x ∈ X such that $x \in F_f$ for all $f \in A_R$ but $x \notin F$. Hence there exists $h \in A_R$ such that $h(x) \neq h(F)$ and therefore, $x \notin F_h$ which is a contradiction. Hence $F = \cap \{ F_f : f \in A_R\}$.

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1.2.4 THEOREM

Let A be a function algebra on X. Then $\mathcal{F}$ $(\widehat{A})$ = { $\tilde{F}$: F ∈ f(A) } and f(A) = $\mathcal{F}$ $(\widehat{A})$ ∩ X = {$\tilde{F}$ ∩ X: $\tilde{F}$ ∈ f$(\widehat{A})$}.

Proof Let F ∈ f(A). First we shall show that $\tilde{F}$ is a set of constancy of $(\widehat{A})_R$. Using [14], it can be checked that $(\widehat{A})_R$ = (A_R)^. Let $\hat{f}$ (A_R)^ where f ∈ A_R. Then $f|_F$ is constant, say α. Let ∅ ∈ $\tilde{F}$. Then there exists a representing measure μ for ∅ which is concentrated on F [14]. Hence $\hat{f}$(∅) = $\int_X$ fdμ =$\int_F$ fdμ = α. Since this is true for any ∅ ∈ $\tilde{F}$, $\hat{f}$ is constant on $\tilde{F}$. Hence $\tilde{F}$ is a set of constancy of $(\widehat{A})_R$. So, $\tilde{F}$ ⊂ K for some K ∈ f$(\widehat{A})$. Therefore, {$\tilde{F}$: F ∈ f(A)} < f$(\widehat{A})$.

Conversely, let G ∈ f($(\widehat{A})$ and G ∩ X = H. Also, let g ∈ A_R . Then $\hat{g}$ ∈ (A_R)^ = $(\widehat{A})_R$ and hence $\hat{g}|_G$ is constant. Therefore, $\hat{g}|_{G\cap X}$ = $g|_{G\cap X}$ = $g|_H$ is constant and so, H is a set of constancy of A_R. Thus H ⊂ K for some K ∈ f(A). But G = (G ∩ X)˜ , as G is a p-set for $\widehat{A}$ [14]. Hence G = $\tilde{H}$ ⊂ $\tilde{K}$ and we get f$(\widehat{A})$ < {$\tilde{F}$: F ∈ f(A)}. Thus f$(\widehat{A})$= {$\tilde{F}$: F ∈ f(A)}.

Since the members of f(A) are p-sets for A, $\tilde{F}$ ∩ X = F for F ∈ f(A) [8]. Hence f(A) = {$\tilde{F}$ ∩ X: $\tilde{F}$ ∈ f$(\widehat{A})$} = $\mathcal{F}$ $(\widehat{A})$ ∩ X.

1.2.5 THEOREM

Let A and B be function algebras on X and Y respectively. Then f(A $\widehat{\otimes}$ B) = f(A) x f(B).

Proof Let F ∈ f(A) and G ∈ (B). Also, let f ∈ $(A \widehat{\otimes} B)_R$ and (x_1, y,), (x_2, y_2) be in F x G. Then f_{x_1} ∈ B_R and f_{y_2} ∈ A_R, where f_{x_1}(y) = f(x_1, y) (y ∈ Y) and f_{y_2}(x) = f(x, y_2) (x ∈ X). So, f_{x_1} is constant on G and f_{y_2} is constant on F. Therefore, f(x_1, y_1) = f_{x_1}(y_1) = f_{x_1}(y_2) = f(x_1, y_2) = f_{y_2}(x_1) = f_{y_2}(x_2) = f(x_2, y_2) and hence f is constant on F x G. Thus f(A) x f(B) < f(A $\widehat{\otimes}$ B).

Conversely, let H ∈ f(A $\widehat{\otimes}$ B). First we show that π_1(H) is a set of constancy of AR, where: X x Y ⟶ X is the projection map. Let g ∈ A_R. Then g $\widehat{\otimes}$ 1 ∈ $(A \widehat{\otimes} B)_R$ and so, g ⊗ 1 is constant on H, i.e., g is constant on π_1(H). Thus π_1(H) is a set of constancy of A_R. Hence π_1(H) ⊂ F for some F ∈ (A). Similarly, we can show that π_2(H) ⊂ G for some G ∈ (B), where π_2: X x Y ⟶Y is the projection map. Thus H ⊂ π_1(H) x π_2(H) ⊂ F x G ∈ (A) x f(B). Hence f(A $\widehat{\otimes}$ B) < f(A) x f(B).

1.2.6 REMARKS

(i) By the same argument as above, one can show that f(A # B) = f(A) x f(B).
(ii) It is clear that $(A \widehat{\otimes} B)_R$, $A_R \widehat{\otimes} B_R$, $(A \# B)_R$ and $A_R \# B_R$ are closed sub-algebras of C_R(X x Y). Also, by the above remark, all these sub-algebras have the same sets of constancy. Hence $(A \widehat{\otimes} B)_R = A_R \widehat{\otimes} B_R = (A \# B)_R = A_R \# B_R$, by [1].

The importance of the Bishop decomposition is due to the Bishop's generalization of the Stone-Weierstrass theorem.

1.2.6 THEOREM

[7] Let A be a function algebra on X and k be the Bishop decomposition for A. If f ∈ (X) and $f|_K \in A|_K$ for every K ∈ k, then f ∈ A.

In our -terminology, it is equivalent to saying that k has the (D)-property for A. Recall that (i)) a decomposition δ of X has the (D)-property for A if f ∈ C(X) and $f_{|_E} \in (A|E)^-$ for every E ∈ δ, then f ∈ A. Since k < f, (ii), f also has the (D)-property for A. On the other hand, if ℓ is a decomposition consisting of closed antisymmetric sets for A, then ℓ may not have the (D)-property for A, as the following example shows.

1.2.8 EXAMPLE

Let D = {z ∈ C : |z| ≤ 1} and S_r = {z ∈ D : |z| = r} for 0 ≤ r ≤ 1. Also, let A = A(P), the disk algebra on the unit disk, i.e., A = {f ∈ C(D) : f is analytic in the interior of D}. Then each S_r, 0 ≤ r ≤ 1, is a set of anti-symmetry for A [14]. Also, ℓ = {S_r 0 ≤ r ≤ 1} is a decomposition of D. Define f : D —-> C by f(z) = |z| . Then f ∈ C(D) and $f|_{S_r} = r \in A|_{S_r}$ for each r, 0 ≤ r ≤ 1. But f ∈ A and hence ℓ does not have the (D>-property for A.

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The essential set E of A and k are related [14]. We shall prove that a similar relation holds between E and f.

1.2.9 PROPOSITION Let P denote the union of all singleton sets in f and E denote the essential set of A. Then E is the closure of X-P.

Proof Let Ṕ denote the union of all singleton sets in k. Then, clearly P⊂ Ṕ and so, X-Ṕ and so ⊂ X-Ṕ. Since E = X-Ṕ [14], E ⊂ $\overline{X-P}$,. Conversely, let x ∈ X-P and if possible, suppose that x ∈ E. Since x ∈ X-P, there exists a nonsingleton member F of f such that x ∈ F. Let y ∈ F, y ≠ x. By Urysohn's lemma, there exists f ∈ C_R (X) such that $f|_E$ ∪{y}= 0 and f(x) = 1. Then f ∈ A_R [14] and f(x) ≠ f(y), which is a contradiction, since x, y e F ∈ f. Hence X-P ⊂ E and since E is closed, we have $\overline{X-P}$ ⊂ E.

We have seen that, in general, k is finer than f and there are examples where k ≠ f. We now give conditions under which k = f.

1.2.10 THEOREM If ℓ is an u.s.c. decomposition of X having the (D)-property for A, then f < ℓ.

Proof. Let X/ℓ denote the quotient space of X obtained by l and q : X →X/ ℓ be the corresponding quotient map. Then, for f ∈ C_R (X/ ℓ), f o q ∈ C_R (X). Also, (f o q)$|_S$ is constant for each S ∈ ℓ. Therefore, (f o q)$|_S$ e A$|_S$ for each S ∈ ℓ Since ℓ has the (D)-property for A, f o q ∈ A. Thus f o q ∈ A_R for each f ∈ C_R(X/ ℓ)

Let F ∈ f and f ∈ C_R(X/ℓ). Then f o q is constant on F, i.e., f is constant on q(F) for every f ∈ C_R (X/ ℓ). Now, X/ ℓ is Hausdorff, as ℓ is u.s.c. [15]. Hence q(F) must be a singleton set and so, F ⊂ S for some S ∈ ℓ, which proves that f < ℓ.

The following corollary is immediate.

1.2.11 COROLLARY [9]. If k is u.s.c., then k = f,

It also follows from the above theorem that the Bishop decomposition determines the Silov decomposition, as the next corollary shows.

1.2.12 COROLLARY

Let A and B be function algebras on X. If k(A) = k(B), then f(A) = f(B)

Proof Suppose k(A) = k(B). Then k(B) is finer than f(A) and hence f(A) has the (D)-property for B. Also, f(A) is u.s.c. . Therefore, - f(B) is finer than f(A). By the same argument, f(A) is finer than - f(B). Hence f(A) = f(B)

The following example shows that the converse of Corollary 3.11 is not true.

1.2.13 EXAMPLE

Let I = [0,1] and D = { z ∈ C : |z| ≤ 1 }, Let X ={ r ,z) ∈ I x D : | z | ≤ $1\frac{r}{2}$ } For a fixed r ∈ I,let X_r = { z ∈ D : (r ,z) ∈ (XX) and r ∈ I, let f_r be defined on x_r by f_r(z) = f(r ,z). Let A = { f ∈ (XX) : f_r A(X_r) for 0 ≤ r ≤ 1} and B = { f (XX) : f_r (X_r) for 0 ≤ r ≤ 1 }. Then A and B are function algebras on X. It can be checked that f(A) = { {x} x x_r : 0 ≤ r ≤ 1} = f(B) = k(B). Further ,k(A) = { {x} x x_r : 0 ≤ r ≤ 1} U { {C0, z} : |z| = } [9]. Hence f(B) = k(A) but k(B) ≠ k(A).

We give some additional results asserting the equality of k and f.

1.2.14 THEOREM

If & has finitely many members, then k = f.

Proof Suppose that f = {F_1, F_2,...,F_n}. We shall show that each F_i is a set of anti-symmetry for A, which proves the result.

For F_1 and $F_j{}_{A'}^{'j\pm1}$ there exists f_{1j} ∈ A_R such that f_{1j} = 1 on F_1 and f_{1j} = 0 on F_j. Take $f_1 = f_{12}f_{13}..... f_{1n}$. Then $f_1 \in A_R$, $f_1 = 1$ *on* F_1 and f_1 = 0 on F_j for all j ≠ 1. Similarly, for each i

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= 2, 3,..., n; there exists $f_i \in A_R$ such that f_i = 1 on F_j and f_j = 0 on F_j for all j ≠ i.

Let g ∈ A and $g|_{F_i}$, be real-valued. Then $h_i = gf_i$ is in A_R and hence h_i is constant on F_i. But h_i = g on F_i and therefore, g is constant on F_i. Thus F_i is a set of anti-symmetry for A. This completes the proof.

Since k < f, it is immediate from that if k has finitely many members, then k = f. In fact, we shall that even when 9K has countable number of members then also k = f.

1.3 OTHER DECOMPOSITIONS

In this section, we consider decompositions of X associated with a function algebra other than those of Bishop and Silov. We study their interrelations and show that some of these decompositions determine the others. Arenson [16] and Ellis [17] have defined and discussed weakly analytic sets and weakly prime sets for A. We introduce the weakly essential, integral domain and analytic decompositions for A. A function algebra A is called an essential function algebra if X is the essential set of A. A is an integral domain if whenever f, $g \in A$ and f g = 0, then f = 0 or g = 0. A is an analytic function algebra if $f \in A$ and f = 0 on a nonempty open set in X, then f = 0.

We have already considered the Bishop and Silov decompositions in detail. For the sake of completeness and for comparing these decompositions with others, we also define below, along with other concepts, the Silov set and a set of antisymmetry for A.

For a closed subset S of X, let Ag denote the uniform closure of $A|_S$ in C(S).

1.3.1 DEFINITIONS

Let S be a closed subset of X and A be a function algebra on X.

(1) S is said to be a weakly essential set for A if Ag is an essential function algebra.
(2) S is said to be a Silov set for A if it is a set of constancy of A_R.
(3) S is said to be a set of antisymmetry for A, if whenever $f \in A$ and $f|_S$ is real-valued, then $f|_S$ is constant.
(4) S is said to be a weakly prime set for A if G is a peak set for A_S implies either G = S or the interior of G, G° = ∅ in the peak set topology (i.e., there is no peak set H, other than S, such that G ∪ H = S).
(5) S is said to be a weakly analytic set for A if G is a peak set for A_S, then either G = S or the interior of G, G° = ∅.
(6) S is said to be an integral domain set Ci.d. set) for A if Ag is an integral domain.
(7) S is said to be an analytic set for A if Ag is an analytic algebra.
It is clear that for the disk algebra A(D), D is an analytic set.

1.3.2 PROPOSITION

Each of the above type of sets is contained in a maximal one of the same type.

Proof First we prove that a weakly essential set is contained in a maximal weakly essential set. Let F be a weakly essential set for A and $\{S_\alpha : \alpha \in \wedge\}$ be the collection of all weakly essential sets for A which contain F. Also, let $S = \overline{\cup_{\alpha \in \wedge} S_\alpha}$. To show that A_S is an essential algebra, let I be a closed ideal of C(S) contained in A_S. Then $\left(I|_{S_\alpha}\right)^-$ is a closed ideal contained in A_{S_α} and hence either $I|_{S_\alpha} = \{0\}$ or $\left(I|_{S_\alpha}\right)^- = C(S_\alpha)$, for each $\alpha \in A$. Suppose for some α, $I|_{S_\alpha} = \{0\}$ and for some β, $\left(I|_{S_\beta}\right)^- = C(S_\beta)$. Then $I|_F = \{0\}$ and also, $(I|_F)^- = C(F)$, since $F \subset S_\alpha \cap S_\beta$. That is, C(F) = {0} which is not possible. Hence for all α, either $I|_{S_\alpha} = \{0\}$ or $\left(I|_{S_\alpha}\right)^- = C(S_\alpha)$. If $I|_{S_\alpha} = \{0\}$ for all α, then I = {0}. Now, suppose $\left(I|_{S_\alpha}\right)^- = C(S_\alpha)$ for all α. Then $C(S_\alpha) = A_{S_\alpha}$, as $\left(I|_{S_\alpha}\right)^- \subset A_{S_\alpha}$. Since A_{S_α} is an essential observed that δ_5 has the (D)-property for A. Imitating the proof given by Ellis [12, Theorem 1] for weakly prime sets, we prove that δ_5 has actually the (GA)-property for A.

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1.3.3 THEOREM

Let A be a function algebra on X. Then δ_5 has the (GA)-property for A. Consequently, each δ_i has the (GA)-property for A, i = 1,2,3,4.

Proof Let $\mu \in b(A^{\perp})^e$ and S = supp μ. It is enough to show that S is a weakly analytic set for A. Let G be a peak set for A_S and suppose that G°≠ $\emptyset$. Then there is a closed set H such that G ∪ H = S. Let $\mu_1 = \mu|_G$ and $\mu_1 = \mu - \mu_1$. Then μ_1 and hence μ_2 are in $A_S^{\perp}$. Also, $\|\mu_1\| + \|\mu_2\| = \|\mu\|$, since μ_1 and μ_2 are singular measures. Suppose $\|\mu_1\| \neq 0 \neq \|\mu_2\|$. Then $\mu = \|\mu_1\| \frac{\mu_1}{\|\mu_1\|} + \|\mu_2\| \frac{\mu_2}{\|\mu_2\|}$ *and* $\frac{\mu_1}{\|\mu_1\|}, \frac{\mu_2}{\|\mu_2\|} \in b(A^{\perp})$ *with* $\|\mu_1\| + \|\mu_2\| = 1$ which is contradiction, since $\mu \in b(A^{\perp})^e$. Therefore, $\mu_1 = 0$ or $\mu_2 = 0$, i.e., S = H or G = S. But H ≠ S, as G°≠ $\emptyset$. Hence G = S and S is a weakly analytic set for A.

1.3.4 REMARK

Since the (GA)-property implies the (S)-property and the (D)-property, it also follows that each δ_i, $i \leq 5$, has the (S)-property and the (D)-property for A.
The following example shows that δ_7 does not have even the CD}-property for A.

1.3.5 EXAMPLE

Consider the function algebra A (ii), i.e., A = A(D)|$_S$ with S = T ∪{0}. Then we have seen that δ_7 = {T, {0}}. Now, define a function g: S —> C by g(z) = |z|. Then $g \in C(S)$, g(0) = 0 ∈ A||$_{\{0\}}$ and g|$_T$ = 1 ∈A|$_T$, . But g ∈ A. Therefore, δ_7 does not have the CD)-property for A.

1.3.6 REMARK

It is proved in [14] that if a maximal function algebra A is essential, then A is analytic. Hence, by Proposition 1.3.1, if A is a maximal function algebra, then all δ_i' s coincide, i.e., δ_7 = δ_6= δ_5 = δ_4 = δ_3 = δ_2 = δ_1.

The results are true for all δ_i's.

1.3.7 PROPOSITION

Let S be a CR set for a function algebra A on X. Then δ_i(A|$_S$) < δ_i(A|$_S$) ∩ S for i ≤ 7.

Proof We have proved the result for i = 2 and 3. Fix i. First we shall show that $f_i(A|_S) < f_i(A) \cap S$. Let F ∈ $f_i(A|_S)$. Since F ⊂ S, it is enough to show that F is an (1)-set for A. Now, F ∈ $f_i(A|_S)$ implies that $\left((A|_S)|_F\right)^-$ is an (i)-algebra, i.e., $(A|_F)^-$ is an (i)-algebra and therefore, F is an (i)-set for A. Consequently, $f_i(A|_S) < f_i(A) \cap S$. Now, $\delta_i(A) \cap S$ is a decomposition of S. Also, if F ∈ $f_i(A|_S)$, then F ⊂ K ∩ S for some K∈ $f_i(A)$. By the construction of δ_i(A), K ⊂ E for some E ∈ $\delta_i(A)$. Therefore, F ⊂ E ⊂ S with E ∈ $\delta_i(A)$. But $\delta_i(A|_S)$ is the finest decomposition of S with such property. Hence $\delta_i(A|_S) < \delta_i(A) \cap S$.

1.3.8 PROPOSITION

Let S be a CR set for A which is saturated with $\delta_i(A)$. Then $\delta_i(A|_S) = \delta_i(A) \cap S$ for i ≠ 2.

Proof By Proposition 1.3.7, $\delta_i(A|_S) < \delta_i(A) \cap S$. Fix i, i ≠ 2. Since S is saturated with $\delta_i(A)$, S is saturated with $f_i(A)$ also. Therefore, $f_i(A) \cap S$ = {F $\in f_i(A): F \cap S \neq \emptyset$. Let $F \in f_i(A) \cap S$, i.e., $F \in f_i(A)$ and F ⊂ S. Then $(A|_F)^-$ is an (i)-algebra. But $\left((A|_S)|_F\right)^- = (A|_F)^-$ and therefore, F is an (i)-set for $A|_S$. Hence $f_i(A) \cap S < f_i(A|_S)$.

Suppose $\delta_i(A|_S) \leqq \delta_i(A) \cap S$. Let $\ell_i = \delta_i(A|_S) \cup$ {E ∈ $\delta_i(A)$: E ∩ S = $\emptyset$}. Then ℓ_i is a decomposition of X and $\ell_i < \delta_i(A)$. Let $F \in f_i(A)$. Then either $F \subset S$ *or* $F \cap S = \emptyset$. If $F \subset S$, then $F \in f_i(A|_S)$ and so, $F \subset G$ for some G ∈ $\delta_i(A|_S)$. If $F \cap S = \emptyset$, then $F \subset E$ for some E ∈ $\delta_i(A)$ with $\delta \cap S = \emptyset$, since S is saturated with δ_i (A). In either case, F is contained in some member of Jand hence X.(A) -<. Therefore, $\ell_i = \delta_i(A)$ and $\delta_i(A|_S) = \delta_i(A) \cap S$.

We have seen that the above result is not true for i = 2.

Let S be a CR set for A and I$_S$ ={$g \in C(X): g|_S = 0$}. Then A+I$_S$ = {f + g : $f \in A$, $g \in I_S$}

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is a function algebra on X [18]. In fact, A+I$_S$ = $\{f \in C(X): f|_S \in A|_S$. Hence it is natural to expect that the decompositions for A + I$_S$ should be related with the corresponding decompositions for A|$_S$ and this does happen as we shall prove now.

1.3.9 PROPOSITION

Let S be a CR set for A. Then $\delta_i(A + I_S) = \delta_i(A|_S) \cup \{\{x\}: x \notin S\}$ for 1 ≤ i ≤ 7.

Proof Since the essential set of A + I$_S$ is the essential set of A|$_S$ [18, Proposition 1.2.1], it is clear that $\delta_1(A + I_S) = \delta_1(A|_S) \cup \{\{x\}: x \notin S\}$ Also, $\delta_2(A + I_S) < \delta_1(A + I_S)$ and so, {{x}:x $\notin S\}$. Let F ⊂ S. Then it is enough to show that $F \in \delta_2(A|_S)$ if and only if $F \in \delta_2(A + I_S)$. But that is true by Proposition 1.3.8, since $\delta_2 = f$. Hence $\delta_2(A + I_S) = \delta_2(A|_S)$\{{x}: x $\notin S\}$

Fix i (i ≥ 3). Since $\delta_1(A + I_S) < \delta_2(A + I_S)$, {{x}:x $\neq S\} \in \delta_i(A + I_S)$. Let F ⊂ S. Then F is an (i)-set for A + I$_S$ if and only if F is an (i)-set for A|$_S$, as (A + I$_S$)$|_S$ = A|$_S$ and F ⊂ S. Hence $F \in f_i(A + I_S)$ if and only if $F \in f_i(A|_S)$ and it can be proved that $\delta_i(A + I_S) = \delta_i(A|_S) \cup \{\{x\}: x \notin S\}$.

1.3.10 DEFINITIONS

(i) A decomposition δ of X is said to be of the first type if there exists only one nontrivial member in δ.

For example, the Bishop decomposition for an antisymmetric algebra is of the first type.

(ii) Let 1 ≤ i ≤ 7. If the decomposition δ_i (A) is of the first type, then the function algebra A is called an almost (i)-algebra.

The above definitions generalize the definitions given by Tomiyama [19, Definition 1.3.10] for an antisymmetric decomposition.

1.3.11 REMARKS

(i) By Proposition 1.3.9, if S is an (i)-set for A which is also a CR set for A, then A + Ig is an almost (i) -algebra, for 1 ≤ i ≤ 7.

(ii) By Proposition 1.3.3, every function algebra is an almost weakly essential algebra.

(iii) By Remark 1.3.6, a maximal function algebra is always an almost (i)-algebra, for 1 ≤ i ≤ 7.

Now, we discuss the decompositions δ_i for the tensor product A $\widehat{\otimes}$ B of function algebras A and B on X and Y respectively.

1.3.12 THEOREM

Let A and B be function algebras on X and Y respectively. Then $f_i = (\mathrm{A} \ \widehat{\otimes} \ \mathrm{B}) = f_i(\mathrm{A}) \ X \ f_i(B) for \ 2 \le i \le 7$.

Proof We have seen that $f_i(\mathrm{A} \ \widehat{\otimes} \ \mathrm{B}) = f_i(\mathrm{A}) \ X \ f_i(B)$ for i = 2 and 3 (iii)). Let i ≥ 4. To prove the required result, it is enough to prove the following:

[a_i] If S and T are (i)-sets for A and B respectively, then S x T is an (i) - set for A $\widehat{\otimes}$ B, i.e. , if $(A_{|S})^-$ and $(B_{|T})^-$ are (i)-algebras, then $(\mathrm{A} \ \widehat{\otimes} \ \mathrm{B}|_{\mathrm{SXT}})^-$ is an -algebra. Since $(\mathrm{A} \ \widehat{\otimes} \ \mathrm{B}|_{\mathrm{SXT}})^-$ = $(A_{|S})^- \ \widehat{\otimes} \ (B_{|T})^-$, we shall prove that if A and B are (i)-algebras, then A $\widehat{\otimes}$ B is an (i)-algebra.

[b_i] If G is an (i)-set for A $\widehat{\otimes}$ B, then π_1(G) and π_2(G) are (i)-sets for A and B respectively, where π_1 : X x Y --► X and π_2 : X x Y —► Y are projection maps.

Case i = 4 Let A and B be weakly prime algebras and E be a peak set for A $\widehat{\otimes}$ B. Suppose that E°≠ $\emptyset$ in the peak set topology, i.e., there is a peak set F, F ≠ X x Y, for A $\widehat{\otimes}$ B such that E ∪ F = X x Y. Let f and g be peaking functions for F and E respectively and G = (X x Y)-F. Since G is open in X x Y, there are open sets U ⊂ X and V ⊂ Y such that U x V ⊂ G. Also, g = 1 on U x V, as U x V ⊂ G ⊂ E. Fix $y_0 \in V$. Then g_{y_o} = 1 on U and therefore, $U \subset \{x \in X: g_{y_o}(x) = 1\} = S \subset X$. Now, if x ∉ S, then g_{y_o} (x) = g(x, y$_o$) ≠1 and therefore, (x, y_0) ∉ E, i.e.,

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(x, y_0) ∈ F and so, f_{y_o} (x) = 1. Thus X-S ⊂ {x ∈ X : f_{y_o} (x) = 1} = T ⊂ X. Also, S and T are peak sets for A and S ∪ T = X. Since A is a weakly prime algebra, S = X or T = X. Suppose S = X. Then g_{y_o} = 1 on X or g = 1 on X x {y_o}, So, g = 1 on X x V, as y_o was an arbitrary point of V. Let (x, y) ∈ X x Y. Then g_x = 1 on V. By the same argument as above, V ⊂ {y' ∈ Y : g_x (y') = 1} = Z ⊂ Y, Y-Z ⊂ {y' ∈ Y : f_x (y') = l}-=W ⊂ Y, Z ∪ W = Y and Z, W are peak sets for B. Since B is a weakly prime algebra, Z = Y or W = Y. If W = Y, then f_x(y) = f(x, y) = 1, i.e., f = 1 on X x Y or F = X x Y which is a contradiction. Therefore, Z = Y and so, g(x, y) = 1, i.e., g = 1 on X x Y or E = X x Y. Similarly, if T = X, then we can show that E = X x Y which shows that A $\widehat{\otimes}$ B is a weakly prime algebra and this completes the proof of [a_4].

To prove [b_4], let G be a weakly prime set for A $\widehat{\otimes}$ B and S be a peak set for $\left(A|_{\pi_1(G)}\right)^-$ with a peaking function f for S. Then (f ⊗ 1))$|_G$ ∈ (A $\widehat{\otimes}$ B$|_G$)$^-$ and it can be checked that (f ® 1)$|_G$ is a peaking function for (S x Y) ∩ G, i.e., (S x Y) ∩ G is a peak set for (A $\widehat{\otimes}$ B$|_G$)$^-$ which is a weakly prime algebra. Therefore, either (S x Y) ∩ G = G or ((S x Y) ∩ G)° = ∅ in the peak set topology. If (S x Y) ∩ G = G, then f ⊗ 1 = 1 on G, i.e., f = 1 on π_1(G) and so, π_1(G) = S. Now, if S°≠ ∅ in the peak set topology, then there is a peak set T, T ≠ π_1(G), for $\left(A|_{\pi_1(G)}\right)^-$ such that S ∪ T = π_1(G). But then (T x Y) ∩ G ≠ G: (T x Y) ∩ G is a peak set for (A $\widehat{\otimes}$ B$|_G$)$^-$ and ((S x Y) ∩ G) ∪ ((T x Y) ∩ G) = G, i.e. , ((S x Y) ∩ G)°≠ ∅ in the peak set topology. Thus if ((S x Y) ∩ G)° = ∅ in the peak set topology, then S° = ∅ in the peak set topology. Therefore, we have either S = π_1(G) or S° = in the peak set topology and hence π_1(G) is a weakly prime set for A. Similarly, we can prove that π_2(G) is a weakly prime set for B.

Case i = 5. By the same argument, we can prove [a_5] and [b_5].

Case i = 6. Suppose that A and B are integral domains. Let f, g ∈ A $\widehat{\otimes}$ B and f ≠ 0 ≠ g. Then there exist (x, y) and (r, s) in X x Y such that f(x, y) ≠ 0 and g(r, s) ≠ 0, i.e., $f_y \neq 0$ and g_s ≠ 0. Also, f_y and g_s are in A which is an integral domain. Therefore, $f_y g_s$ ≠ 0 and so, $(f_y g_s)(p)$ ≠ 0 for some point p ∈ X. Thus $f_p(y) = f(p,y) = f_y(p) \neq 0$ and $g_p(s) = g(p,s) = g_s(p) \neq 0$. Since f_p and g_p are in B and B is an integral domain, $f_p g_p$ ≠ 0, i.e., for some q ∈ Y, $(f_p g_p)(q) \neq 0$ or $(fg)(p,q) \neq 0$. Hence fg ≠ 0 and so, A $\widehat{\otimes}$ B is an integral domain which proves [a_6].

To prove [b_6], let G be an i.d. set for A $\widehat{\otimes}$ B and f, $g \in \left(A|_{\pi_1(G)}\right)^-$ such that f ≠ 0 ≠ g. Then (f ⊗ 1)$|_G$ and $(g \otimes 1)|_G$ are in (A $\widehat{\otimes}$ B$|_G$)$^-$ and it is clear that $(f \otimes 1)|_G \neq (g \otimes 1)|_G$. Since (A $\widehat{\otimes}$ B$|_G$)$^-$ is an integral domain, $(f \otimes 1)(g \otimes 1) \neq 0$ on G and so, fg ≠ 0. Hence π_1(G) is an i.d. set for A. Similarly, π_2(G) is an i.d. set for B.

Case i = 7. Finally, we prove [a_7] and [b_7]. Assume that A and B are analytic algebras. Let h ∈ A $\widehat{\otimes}$ B be zero on a nonempty open subset G of X x Y. There are nonempty open sets U ⊂ X and V ⊂ Y such that U x V ⊂ G. Fix $y_o \in V$. Then h_{y_o} = 0 on U which is a nonempty open subset of X. But $h_{y_o} \in A$ which is analytic and hence h_{y_o} = 0 on X, i.e., h = 0 on X x {y_o}. Since y_o is an arbitrary point of V, h=0 on X x V. Now, let (x, y) ∈ X x Y. Then $h_x \in B$ and h_x = 0 on V. Thus h_x = 0 on Y, as $h_x \in B$ and B is analytic, i.e. , h(x, y) = h_x(y) = 0. Hence h = 0 on X x Y and consequently, A $\widehat{\otimes}$ B is an analytic algebra.

To prove [b_7], let G be an analytic set for A $\widehat{\otimes}$ B and $f \in \left(A|_{\pi_1(G)}\right)^-$be zero on a nonempty open set U of π_1(G). Then (U x Y) ∩ G is open in G, (U x Y) ∩ G ≠ ∅ and $(f \otimes 1)|_G$ ∈ $\left(A \widehat{\otimes} B|_G\right)^-$ is zero on (U x Y) ∩ G. Hence $(f \otimes 1)|_G$ = 0, since $\left(A \widehat{\otimes} B|_G\right)^-$ an analytic algebra. Thus f = 0 on π_1(G) and so, π_1(G) is an analytic set for A which completes the proof of the theorem.

1.3.13 REMARKS

(i) It can be shown that for 2 ≤ i ≤ 7, $f_i(A)$ x $f_i(B) = f_i(A \widehat{\otimes} B)$, then $\delta_i\left(A \widehat{\otimes} B\right) < \delta_i(A) < \delta_i(B)$. Hence, in view of Theorem 1.3.15, we always have $\delta_i\left(A \widehat{\otimes} B\right) < \delta_i(A) < \delta_i(B)$ for 2 ≤ i ≤ 7.

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(ii) It is shown in [18] that the essential set of $A \widehat{\otimes} B$ is (E_A x Y) ∪ (X x E_B), where E_A and E_B denote the essential sets of A and B respectively. Thus $\delta_i(A \widehat{\otimes} B)$ = {(E_A x Y) ∪ (X x E_B) U {(x, y) : x ∉ E_A , y ∉ E_B}. But $\delta_1(A)$ x $\delta_1(B)$ = {E_A x E_B} U ({x} x E_B : x ∉ E_B} U {E_A x {y} : y ∉ E_B} U {(x, y)}: x ∉ E_A, y ∉ E_B}. Hence $\delta_1(A)$ x $\delta_1(B) \lneqq \delta_1(A \widehat{\otimes} B)$, if E_A ≠ X or E_B ≠ Y.

If A is a function algebra on X, then $\hat{A}$, the Gelfand transform of A, is a function algebra on m(A). To investigate the relation between decompositions of X and of m(A), we need the following proposition due to Hayashi [8, Theorem 1.5]. We recall that a JL closed subset E of X is a p-set for A if $\pi \in A^{\perp}$ implies that $\mu_E \in A^{\perp}$.

1.3.14 PROPOSITION

There is a one-to-one correspondence between peak sets (p-sets) E for A and peak sets (p-sets) F for $\hat{A}$ such that $\tilde{E} = F$ and F ∩ X = E. Further,$(\bigcap_\alpha E_\alpha)^{\sim} = \bigcap_\alpha E_\alpha^{\sim}$ where $E_\alpha$is a peak set Cp-set} for A for each α.

1.3.15 THEOREM

A function algebra A is an (i)-algebra on X if and only if $\hat{A}$ is an (i)-algebra on m(A), for i = 1,2,3,4 and 6.

Proof It is well known that the essential set $E_{\hat{A}}$ of $\hat{A}$ is E_A ∪ (m(A)-X), where E_A denotes the essential set of A [14]. Therefore, E_A = X if and only if $E_{\hat{A}}$= m(A), i.e., A is an essential algebra if and only if $\hat{A}$ is an essential algebra or equivalently A is a weakly essential algebra if and only if $\hat{A}$ is a weakly essential algebra.

By Theorem 1.2.5 and Theorem 1.2.3(ii), A is an (i)-algebra if and only if $\hat{A}$ is an (i)-algebra, for i = 2 and 3.

Let i = 4. By Proposition 1.3.14, a subset S of X is a peak set for A if and only if $\tilde{S}$ is a peak set for $\hat{A}$. Also, S = X if and only if $\tilde{S}$ = m(A), since $\tilde{S}$ ∩ X = S. So, to prove that A is weakly prime if and only if $\hat{A}$ is weakly prime, it is enough to show that S° = ∅ in the peak set topology if and only if $(\tilde{S})$° = ∅ in the peak set topology. Suppose that $(\tilde{S})$°≠ ∅ in the peak set topology. Then there is a peak set T, T ≠ X, for A such that $\tilde{S} \cup \tilde{T}$ = m(A). But then S ∪ T = X. Thus S°≠ ∅ in the peak set topology. Similarly, we can show that if S°≠ ∅ in the peak set topology, then $(\tilde{S})$° ≠ ∅ in the peak set topology.

Let A be an integral domain and $\hat{f}$, $\hat{g} \in \hat{A}$ with $\hat{f} \neq 0 \neq \hat{g}$, where f, g ∈ A. Then f ≠ 0 ≠ g, since A $||\hat{f}||$ = ||f||. But A is an integral domain and so, fg ≠ 0 Therefore, $(fg)^\wedge \neq 0$, i.e., $\hat{f}\hat{g} \neq 0$ and hence A is an integral domain. Conversely, assume that $\hat{A}$ is an integral domain and f, g ∈ A such that f ≠0 ≠g. Then $\hat{f}$, $\hat{g} \in \hat{A}$ and $\hat{f} \neq 0 \neq \hat{g}$ So, $\hat{f}\hat{g} \neq 0$, i.e., $(fg)^\wedge \neq 0$ and therefore, fg ≠ 0. Hence A is an integral domain.

1.3.16 COROLLARY

Let A be a function algebra on X. Suppose that the members of $f_i(\hat{A})$ are p-sets for $\hat{A}$ for i = 4 and 6. Then $f_i(\hat{A})$ = {$\tilde{S}$: S ∈ $f_i(A)$} and $f_i(A) = f_i(\hat{A})$ ∩ X = {$\tilde{S}$ ∩ X : $\tilde{S} \in f_i(\hat{A})$} for i = 1, 2, 3, 4 and 6.

Proof Since $E_{\hat{A}}$ = E_A ∪ (m(A)-X) is a p-set for $\hat{A}$, $(E_{\hat{A}} \cap X)^{\sim} = E_{\hat{A}}$ i.e., $\tilde{E_{\hat{A}}} = E_{\hat{A}}$. Also, if x ∈ X, then x ∉ E_A if and only if x ∉ $E_{\hat{A}}$. Therefore, $\delta_i(\hat{A}) = f_1(\hat{A})$= {$\tilde{E_{\hat{A}}}$} U {{x} : x x ∉ E_A }.

By Theorem 1.2.5 and Theorem 1.2.3(ii), the result is true for i = 2 and 3. Let i = 4 or 6. Then S is an (i)-set for A iff $(A_{|S})^{-}$ is an (i)-algebra iff $((A_{|S})^{-})^{\wedge}$ is an (i)-algebra iff $(\hat{A}|_{\tilde{S}})^{-}$ is an (i)-algebra, (since $((A_{|S})^{-})^{\wedge}$ = $((A_{|S})^{\wedge})^{-}$ = $(\hat{A}|_{\tilde{S}})^{-}$) iff $\tilde{S}$ is an (i)-set for $\hat{A}$.

Let $S \in f_i(A)$. Then $\tilde{S}$ ⊂ H for some H ∈ $f_i(\hat{A})$. By our assumption H is a p-set for $\hat{A}$. Hence, by Proposition 1.3.17, H = $\tilde{G}$, where G = H ∩ X is a p-set for A. Now, S ⊂ $\tilde{S}$ ∩ X ⊂ H ∩ X = G which is an (i)-set for A, as $\tilde{G} \in f_i(\hat{A})$, Therefore: S = G and $\tilde{S} \in f_i(A)$. Conversely, let $\tilde{G}$

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$\in f_i(\hat{A})$ for some p-set G for A. Then G is an (i)-set for A and so, G ⊂ S for some S ∈ $f_i(A)$. But then $\tilde{S}$ is an (i)-set for $\hat{A}$ and $\tilde{G} \subset \tilde{S}$. Therefore, $\tilde{G} = \tilde{S}$ and G = $\tilde{G}$ ∩ X = $\tilde{S}$ ∩ X. Thus G ⊂ S ⊂ $\tilde{S}$ ∩ X = G, i.e., G =S ∈ $f_i(A)$. Hence $f_i(\hat{A})$ = {$\tilde{S}$: S ∈ $f_i(A)$ } and ∈ $f_i(A)$ = $f_i(\hat{A})$ ∩ X. Since every $\tilde{S} \in f_i(\hat{A})$ intersects X, $f_i(\hat{A})$ n X = {$\tilde{S}$ n X : $\tilde{S} \in f_i(\hat{A})$}.

1.3.17 REMARK

Let i = 4 or 6. Suppose that members of δ_i(A) and $f_i(\hat{A})$ are p-sets for A and $\hat{A}$ respectively. Since $f_i(A) = f_i(\hat{A})$ ∩ X, δ_i(A) < $f_i(\hat{A})$ ∩ X. Let $(\delta_i(A))^{\sim}$ = {$E_i^{\sim}$: E_i ∈ δ_i(A). Since members of δ_i(A) are p-sets for A, $(\delta_i(A))^{\sim}$ is a decomposition of m(A). For S_i ∈ f_i(A), $S_i^{\sim} \subset E_i^{\sim}$ for some E_i ∈ δ_i(A). But $S_i^{\sim} \in f_i(\hat{A})$ and therefore, $\delta_i(\hat{A})$ < $(\delta_i(A))^{\sim}$. Thus $\delta_i(\hat{A})$ ∩ X < $(\delta_i(A))^{\sim}$ ∩ X = δ_i(A). Hence δ_i(A) = $\delta_i(\hat{A})$ ∩ X.

We do not know whether A is weakly analytic implies that $\hat{A}$ is weakly analytic or $\hat{A}$ is weakly analytic implies that A is weakly analytic.

1.3.18 EXAMPLE

Consider the function algebra A = A(D)|S of Example 1.3.5(ii). Then we know that δ_7 = {T, {0}} and hence A is not analytic. However, $\hat{A}$ = A(D) which is analytic. For an example of a function algebra A which is analytic but $\hat{A}$ is not analytic, we refer to [20, MR 81j, 46083],

In section 1, we have proved that the Bishop decomposition determines the Silov decomposition (Corollary 1.2.13). We prove that some of the decompositions δ_i will determine the others. We use the following ideas of Sidney [11].

Let A be a function algebra on X and f(A) denote the Silov decomposition for A. We denote an ordinal number by σ. Define inductively, the decompositions $\ell_\sigma = \ell_\sigma(A)$ of X into closed subsets as follows:
(i) ℓ_o = {X};
(ii) $\ell_{\sigma+1} = \{F : F \in f(A|_E), E \in \ell_\sigma >$ and
(iii) if σ is a limit ordinal, then $\ell_\sigma = \{E_\sigma : E_\sigma = \cap_{\sigma,<\sigma} E_{\sigma'} , E_\sigma \in \ell_\sigma\}$ i

Sidney [11] has observed the following results: (S_1) The above inductive process of taking decompositions terminates at some point. Let σ(A) denote the first ordinal number σ such that $\ell_{\sigma+1} = \ell_\sigma$.
(S_2) $\ell_{\sigma(A)} = k(A)$, the Bishop decomposition for A.
(S_3) Each E ∈ ℓ_σ is a p-set for A and it is the union of members of k(A) for every ordinal σ.
Now, we are ready to prove the result regarding the determination of one type of decomposition by the other.

1.3.19 PROPOSITION

Let A and B be function algebras on X. Then δ_4(A) = δ_4 (B) ⇒ δ_3(A) = δ_3(B) ⇒ δ_2(A) = δ_2(B) ⇒ δ_1(A) = δ_1(B) and δ_5 (A) = δ_5 (B) ⇒ δ_3 (A) = δ_3(B).

Proof. Suppose δ_2(A) = δ_2(B). It is enough to show that $E_A = E_B$, where $E_A(E_B)$ denote the essential set of A(B). Let P_2(A) and P_2(B) denote the set of all singleton elements of δ_2(A) and δ_2(B) respectively. Then $E_A = \overline{X - P_2(A)}$ and $E_B = \overline{X - P_2(B)}$, by Proposition 1.2.10. Since δ_2(A) = δ_2(B), we have P_2(A) = P_2(B). Therefore, $E_A = E_B$ and hence δ_2(A) = δ_2(B).

We have shown that δ_3(A) = δ_3(B) ⇒ δ_2(A) = δ_2(B) (Corollary 1.2.13),

Assume that δ_4(A) = δ_4 (B). First we shall show that δ_2(A) = δ_2(B). For this, it suffices to show that $A_R = B_R$. Let f ∈ A_R. Then $f|_F$ is constant for each F ∈ δ_2(A), Since δ_4 (B) = δ_4 (A) < δ_2 (A), $f|_H$ is constant for each H ∈ δ_4 (B). So, $f|_H \in B|_H$ for each H ∈ δ_4(B). Since δ_4(B) has the (D)-property for B, f ∈ B. Thus $A_R \subset B_R$. Similarly, $B_R \subset A_R$ and hence δ_2(A) = δ_2(B). Now, we use Sidney's technique. Accordingly, δ_2(A) = δ_2(B) is equivalent to saying that $\ell_1(A) = \ell_1(B)$. Suppose $\ell_\sigma(A) = \ell_\sigma(B)$. We want to show that $\ell_{\sigma+1}(A) = \ell_{\sigma+1}(B)$. Since $\ell_{\sigma+1}(A)$ = {F ∈ $f(A|_E)$: E ∈ $\ell_\sigma(A)$}, it is enough to show that $(A|_E)_R = (B|_E)_R$ for E ∈ $\ell_\sigma(A) = \ell_\sigma(B)$. Let E ∈ $\ell_\sigma(A)$ and f ∈ $(A|_E)_R$. By (S_3), E is saturated with k(B) and hence with δ_4(B), since δ_4(B) < δ_3(B) = k(B). Let H ∈ δ_4(B) = δ_4(A) and H ⊂ E. Since δ_4(A) < δ_3(A) and (S_3) holds, there

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exists K ∈ δ_3(A) such that H ⊂ K ⊂ E. It follows that $f|_H$ is constant. Thus f ∈ C(E) and $f|_H \in (B|_E)|_H$ for each H ∈ δ_4(B), H ⊂ E. Also, by (S_3), E is a p-set for B. Since δ_4(B) has the (S)-property for B, $\delta_4(B) \cap$ E has the (D)-property for $B|_E$ and hence f ∈ $(B|_E)_R$ and $(A|_E)_R \subset (B|_E)_R$. Similarly, $(B|_E)_R \subset (A|_E)_R$. and so $\ell_{\sigma+1}(A) = \ell_{\sigma+1}(B)$. Let σ' be limit ordinal and suppose that $\ell_\sigma(A) = \ell_\sigma(B)$ for all $\sigma = \sigma'$. Then E ∈ $\ell_{\sigma'}(A)$ iff E = $\bigcap_{\sigma<\sigma'}\{E_\sigma : E_\sigma \in \ell_\sigma(A)\}$ iff E = $\bigcap_{\sigma<\sigma'}\{E_\sigma : E_\sigma \in \ell_\sigma(B)\}$ iff E ∈ $\ell_{\sigma'}(B)$. Hence, for each ordinal σ, we have E $\ell_\sigma(A) = \ell_\sigma(B)$. So, δ_3(A) = δ_3(B).

By the same argument, we can prove that δ_5(A) = δ_5(B) => δ_3(A) = δ_3(B).

1.3.20 REMARK

The decomposition δ_7 does not determine any other decomposition. For, let A = $A(D)|_S$ of Example 4.4.5(ii) and B = A + I_T . Then δ_7(A) = {T, {0}} and δ_i(A) = {S} for each i < 7. Also, δ_i(B) = δ_i $(A|_T)$ U {x} x ∉ T for all i ≤ 7. Therefore, δ_i(B) = {T, {0}} for i ≤ 7. Hence δ_7(A) = $\delta_7(B)$ = {T, {0}}. But δ_i(B) = {T, {0}} < {S} =$\delta_i(A)$ for i < 7.

1.4 CONCLUSIONS

In this paper, we deal with certain decompositions of a compact Hausdorff space X associated with a function algebra A. The most well known decompositions are those of Bishop and Silov. While the two decompositions coincide in case of many known function algebras, there are function algebras where the two differ. It is natural to look for conditions under which these decompositions are coincident. Section 1 is devoted to the study of this problem. In section 2, we study the Bishop and Silov decompositions for restriction algebras. The last section is devoted to the study of several other decompositions such as analytic, weakly analytic, weakly prime etc. In addition to proving certain properties of these decompositions, we discuss these decompositions for the tensor product. Finally, we show that some of these decompositions determine some of the others.

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