

DEFICIENT CUBIC SPLINE

Ram Swaroop Shukl, Research Scholar¹

Dr. Alok Kumar, Professor²

Department of Mathematics, Swami Vivekanand University, Sagar, M.P.^{1,2}

ABSTRACT

In the present paper, we shall study existence, uniqueness and convergence property of extrapolated cubic spline with multiple knots which interpolate a given function at two points of a general choice of set of point's interior to each mesh interval which includes some earlier results in this direction of particular choice.

Keywords: - Extrapolated Cubic Spline, Mesh Interval and function

1. INTRODUCTION

Splines are curves, and they are usually required to be continuous and smooth. Splines are usually defined as piecewise polynomials of degree k with function values and first $k-1$ derivative that agree at the points where they join. The abscissa or X -axis values of the join points are called *knots*. The term "spline" is also used for polynomials (splines with no knots) and piecewise polynomials with more than one discontinuous derivative. Splines with no knots are generally smoother than splines with knots, which are generally smoother than splines with multiple discontinuous derivatives. Splines with few knots are generally smoother than splines with many knots; however, increasing the number of knots usually increases the fit of the spline function to the data. Knots give the curve freedom to bend to more closely follow the data (Schumaker [9]).

An important development in this direction is the introduction of discrete spline by Mangasarian and Schumaker [7] (See also Rana and Dubey [8], Dikshit and Rana [6], Astor and Duris [2], and Choudhary, Dubey & Rana [4]). It is mentioned that continuous cubic spline may be used as a limiting case of the discrete cubic spline. In fact the defining condition for discrete cubic spline involves in some sense a certain process of extrapolation. We use this approach here for defining extrapolated

deficient cubic splines. The class of all piecewise polynomial functions s_i of degree 3 or less which satisfy the condition,

$$(s_{i+1} - s_i)(x_i - jh) = 0, i = 1, 2, \dots, n, \tag{1.1}$$

for $h > 0$ and $j = 0, 1$ defines the class $D(3, P, h)$ of extrapolated deficient cubic splines. To be more specific we denote the elements of $D(3, P, h)$ by s^h .

It may be mentioned that condition (1.1) is less stringent the corresponding condition used for defining discrete cubic splines. We shall study in the present paper existence, uniqueness and convergence property of extrapolated cubic spline with multiple knots which interpolate a given function at two points of a general choice of set of points interior to each mesh interval which includes some earlier results in this direction of particular choice.

We set for convenience

$$u_i = x_{i-1} + (1/6)p_i$$

and

$$v_i = x_{i-1} + (5/6)p_i \text{ for } i = 1, 2, \dots, n$$

where $1/6, 5/6$ are real numbers and p_i is the length of mesh interval $[x_{i-1}, x_i]$ for the mesh P of $[0, 1]$ given by $P: 0 = x_0 < x_1 < \dots < x_n = 1$

and $p = \max_i p_i, p' = \min_i p_i$.

We propose to study the following:

Problem 1. Given functional values $\{f(u_i)\}$ and $\{f(v_i)\}$, to find the condition on θ and p which lead to a unique extrapolated deficient cubic splines satisfying the following interpolatory conditions :

$$s^h(u_i) = f(u_i) \tag{1.2}$$

$$s^h(v_i) = f(v_i) \text{ for } i = 1, 2, \dots, n \tag{1.3}$$

2. EXISTENCE AND UNIQUENESS

In order to answer the problem 1, we set for convenience

$$R_i(x) = (x - x_{i-1})(x - u_i)(x - v_i),$$

$$Q_i(x) = (x - x_i)(x - u_i)(x - v_i),$$

$$R_i(x, u) = (x - x_{i-1})(x - u_i)^2$$

$$Q_i(x, v) = (x - x_i)(x - v_i)^2.$$

$R_i(x)$ with the factor $(x - x_{i-1})$ replaced by $(x - x_i)$ define $Q_i(x)$, we state the following equations which will be useful,

$$R_i(x_i - h) = (1/36)(p_i - h)(p_i - 6h)(5p_i - 6h)$$

$$Q_i(x_i - h) = (-h/36)(p_i - 6h)(5p_i - 6h)$$

$$R_i(x_i - h, u) = (1/36)(p_i - h)(5p_i - h)^2$$

$$\text{and } Q_i(x_i - h, v) = -36h(p_i - 6h)^2.$$

We shall answer the problem 1 in the following.

Theorem 2.1: Suppose that f is 1 periodic and p' is such that for $h > 0$ (i) $p' \geq 3h$ or (ii) $\langle p_i \rangle$ is a non-increasing sequence with $p' \geq h$, $I = 1, 2, \dots, n$ holds then there exist a unique 1 periodic spline $s^h \in D(3, P, h)$ which satisfies the interpolatory condition (1.2) and (1.3).

Proof. It is clear that $s^h \in D(3, P, h)$ then we may write

$$S_i^h(x) = A Q_i(x) - B R_i(x) + C R_i(x, u) - D Q_i(x, v) \tag{2.1}$$

where A, B, C and D are constants to be determined.

$$f(u_i) = D(20/54) p_i^3 \tag{2.2}$$

$$\text{and } f(v_i) = C(3/27) P_i^3 \tag{2.3}$$

If we now set $s^h(x_i) = N_i(h)$, $i = 0, 1, 2, \dots, n$ and use (2.2) and (2.3), then we have from (2.1).

$$N_i(h) = 9[-(5/36) P_i^3 B + (5/6) f(v_i)] / 4 \tag{2.4}$$

$$\text{and } N_{i-1}(h) = (9/4)[-(1/36)P_i^3 A + (5/6) f(u_i)] \quad (2.5)$$

Thus in view of (2.2) - (2.5), we see that for the interval $[x_{i-1}, x_i]$,

$$\begin{aligned} (1/81) p_i^3 s_i^h(x) &= (4/9)[R_i(x) N_i h - Q_i(x) N_{i-1}(h)] \\ &+ (1/6) f(u_i)[5Q_i(x) - Q_i(x, v)] + f(v_i)[(1/6)R_i(x, u) - (5/6)R_i(x)]. \end{aligned} \quad (2.6)$$

Now it follows from (2.6) and (2.1) with $j=1$ that

$$\begin{aligned} hL(p_i, -h) p_{i+1}^3 N_{i-1}(h) &+ [(p_i - h)L(p_i, -h) p_{i+1}^3 - (p_{i+1} + h)L(p_{i+1}, h) p_i^3] \\ N_i(h) &+ hL(p_{i+1}, h) p_i^3 N_{i+1}(h) = F_i(1/6, h) \end{aligned} \quad (2.7)$$

Where

$$\begin{aligned} (2/3) F_i(1/6, h) &= (1/36 h p_i^3 (p_{i+1} + h)[p_{i+1} + 6h] f(v_{i+1}) - (5 p_{i+1} + 6h) f(u_{i+1})) \\ &- (1/6) h p_{i+1}^3 (p_i - h)[5 p_i - 6h] f(v_i) - (1/6)(p_i - 6h) f(u_i) \end{aligned}$$

and

$$L(P_i, jh) = (1/36)(p_i + 6jh)(5 p_i + 6jh) \text{ for all } i = j = 1, -1.$$

In order to prove theorem 2.1, it is sufficient to show that the system of equation (2.7) for $I = 1, 2, \dots, n$ has a unique set of solutions. Clearly the coefficients of $N_{i+1}(h)$ is non-negative. Further in view of the condition (ii) of theorem 2.1 as $p_{i+1} / p_i \leq 1$, we observe that the coefficients of $N_i(h)$ is non-positive and the absolute value of the coefficient of $N_{i-1}(h)$ is

$$|hL(P_i, -h) P_{i+1}^3| < hL(P_i, h) p_{i+1}^3 = h(P_i + 6h)(5 p_i + 6h)$$

Thus, the excess of the positive value of the coefficient of $N_i(h)$ over the sum of the positive value of the coefficients of $N_{i-1}(h)$ and $N_{i+1}(h)$ in (2.7) is less than

$$a_i(h) = h p_i p_{i+1} [p_i^2 (p_{i+1} + h) + p_{i+1}^2 (p_i - 3h)]$$

which is clearly positive under the condition (i) or (ii) of Theorem 2.1.

We thus conclude that the coefficient matrix of the system of equation (2.7) is diagonally dominant and hence invertible. This completes the proof of Theorem 2.1.

3. ERROR BOUNDS

In this section of the present paper, we should estimate the bounds for the error function $e = s^h - f$, where s^h is the interpolatory spline of Theorem 2.1. For convenience we assume in this section of this paper, that the mesh points are equidistant, so that

$$p_i = p, \quad i = 0, 1, \dots, n.$$

We now introduce function $t^{(r)}$ which is the same as $(s^h)^{(r)}$. At the mesh points the function $t^{(r)}$, $r=1, 2, \dots$ is defined by

$$t^{(r)}(x_i) = (s_i^h)^{(r)}(x_i), \quad i = 0, 1, \dots, n. \quad (3.1)$$

It is of course clear that since $s^h \in C[0, 1]$

Using the foregoing notation of $t^{(r)}(x)$, we shall prove the following.

Theorem 3.1. Suppose that $f^{(r)}$ exist in $[0, 1]$ then for interpolatory spline s^h of theorem 2.1 we have

$$\|(t^{(r)} - f^{(r)})(x)\| \leq ((5/6)p)^{2-r} K(h, 1/6) w(f^{(r)} p) \quad \text{for } r=0, 1, 2, \dots \quad (3.2)$$

where $K(h, 1/6)$ is a positive function of h .

Proof. It may be observed that the system of equations (2.7) may be written as

$$A(h) N(h) = F(h), \quad (3.3)$$

Where, $A(h)$ is the coefficient matrix having non zero element in each row.

$N(h) = (N_i(h))$ and $F(h)$ denotes the single column matrix $(F_i(\theta, h))$. In view of the diagonal dominant property of $A(h)$ (Ahlberg, Nilson and Walsle [1]). It may be seen that

$$\|A^{-1}(h)\| \leq a(h) \quad (3.4)$$

where $a(h) = \{2h p^4 (p - h)\}^{-1}$

We rewrite the equation (3.4) to obtain

$$A(h)(N_i(h) - f_i) = F_i(1/6, h) - A(h) f_i. \quad (3.5)$$

We first proceed to estimate the right hand side of (3.5). Applying the Taylor's theorem appropriately we observe that the i^{th} row of the right hand side of appearing in (3.5) is

$$\begin{aligned} & (1/162)h p^5 [(p+h)\{25(p+6h)f''(\alpha_{i+1}) - (5p+h)f''(\beta_{i+1})\} \\ & - (p-h)\{(5p-6h)f''(\alpha_i) - 25(p-6h)f''(\beta_i)\}] \\ & - h p^5 (1/27)[(p+6h)(5p+6h)f''(z_i) + (p-6h)(5p-6h)f''(\delta_{i+1})]/2(1-2\theta) \end{aligned}$$

where α_i, β_i, z_i and $\delta_i \in [x_{i-1}, x_i]$ for all i .

Now using (3.4) and adjusting suitably the terms of right hand side of (3.5), we have

$$\|(N_i(h) - f_i)\| \leq p^2 K_1(l, \theta) w(f'', p) \quad (3.6)$$

where

$$K_i(l, 1/5) = (1/2)[5 + (11)d + 2d^2]/h(1-d) \text{ with } d = p/h.$$

Observing that

$$R_i''(x_i) = 2Q_i''(x_i) = hp$$

$$Q_i''(x_i, v) = h p/6$$

and

$$R_i''(x_i, u) = 16p/3,$$

we have from (3.6),

$$\begin{aligned} & (1/54)p^2 [(s_i^h)''(x_i) - f''(x_i)], \\ & = (4/3)[2(N_i(h) - f_i) - (N_{i-1}(h) - f_{i-1})] + G_i(f) \end{aligned} \quad (3.7)$$

Where

$$\begin{aligned} G_i(f) &= (7/3)f(u_i) - (11/3)f(u_i) + (8/3)f_i \\ & - (4/3)f_{i-1} - (1/54)p^2 f''(x_i). \end{aligned}$$

By an appropriate application of Taylor's theorem, we have

$$G_i(f) = p^2 [35/216 f''(\beta_i) - (11/216) f'''(\alpha_i) - (2/3) f''(\delta_i) - (1/54) f''(x_i)]$$

where $\alpha_i, \beta_i, \delta_i \in [x_{i-1}, x_i]$ for all i .

Again adjusting suitably the terms of $G_i(f)$, we get

$$\|G_i(f)\| \leq (115/108) p^2 w(f'', p). \tag{3.8}$$

Then using (3.6) and (3.8), we have from (3.7)

$$\|((s_i^h)'' - f'')(x_i)\| \leq K_2(h, 1/6) w(f'', p)$$

where $K_2(h, 1/6) = 54[4K_1(l, \theta) + (115/108)]$

$(s_i^h)''$ is piecewise linear, so that for $[x_{i-1}, x_i]$,

$$p(s_i^h)''(x) = (s_i^h)''(x_{i-1})(x_i - x) + (s_i^h)''(x_i)(x - x_{i-1}) \tag{3.9}$$

and hence,

$$\begin{aligned} p((s_i^h)'' - f'')(x) &= (x_i - x)[(s_i^h)''(x_{i-1}) - (s_{i-1}^h)''(x_{i-1})] + \\ &+ (x_i - x)[(s_{i-1}^h)''(x_{i-1}) - f_{i-1}''] + (x - x_{i-1})[(s_i^h)''(x_i) - f_i''] \\ &+ (x_i - x)(f_{i-1}'' - f''(x)) + (x - x_{i-1})(f_i'' - f''(x)). \end{aligned}$$

$$\text{Thus, } \|((s_i^h)'' - f'')(x)\| \leq \|(s_i^h)'' - (s_{i-1}^h)''(x_{i-1})\| + \|((s_i^h)'' - f'')(x_i)\| + w(f'', p) \tag{3.10}$$

Next, we see that

$$2R_i''(x_{i-1}) = Q_i''(x_{i-1}) = -hp,$$

$$R_i''(x_{i-1}, u) = -hp/6$$

$$\text{and } Q_i''(x_{i-1}, v) = -16p/3$$

$$\begin{aligned} \text{So in view of (3.6), we have } 5/4 p^2 (S_i^h)''(x_{i-1}) &= (4/3)[2N_{i-1}(h) - N_i(h)] \\ &- (11/3)f(u_i) + (7/3)f(v_i) \end{aligned}$$

$$\text{and } (1/54) p^2 (s_i^h)''(x_i) = (4/3)[2N_i(h) - N_{i-1}(h)] + (7/3)f(u_i) - (11/3)f(v_i).$$

Thus,

$$1/54) p^2 [(s_i^h)'' - (s_{i-1}^h)''](x_{i-1})] = (4/3)[N_{i-2}(h) - f_{i-2}] - (N_i(h) - f_i) + V_i(f) \quad (3.11)$$

Where

$$V_i(f) = (4/3)[f(v_{i-1}) - f(u_i)] + (7/3)[f(v_i) - f(u_{i-1})] + (4/3)[f_{i-2} - f_i]. \quad (3.12)$$

Again using the Taylor's theorem appropriately, we see that,

$$\|V_i(f)\| \leq (265/36) p^2 \cdot w(f'', p) \quad (3.13)$$

and therefore, using (3.6) and (3.13), we have from (3.12)

$$\|((s_i^h)'' - (s_{i-1}^h)'')(x_{i-1})\| \leq K_3(h, 1/6) w(f'', p) \quad (3.14)$$

where $K_3(h, 1/6) = 144[K_1(h, 269/36)]/$

Thus, combining (3.9), (3.11) and (3.14), we get

$$\|((s_i^h)'' - f'')(x)\| \leq (1 + K_2(h, 0)^t + K_3(h, 1/6) w(f'', p)), \quad (3.15)$$

which proves the result of Theorem 3.1 for $r=2$, with

$$K(h, 1/6) = 1 + K_2(h, 1/6) + K_3(h, 1/6).$$

Next, we observe that in view of the interpolatory condition (3.2) and (3.3), there exist a point $t_i \in (u_i, v_i)$ s.t. $(s_i^h)' - f'(t_i) = 0$.

Thus, for any $x \in [x_{i-1}, x_i]$

$$\begin{aligned} \|(t^{(1)} - f')(x)\| &\leq \max_i \left| \int_{Q_i}^x (t^{(2)} - f'')(q) dq \right| \\ &\leq \theta^* p \|(t^{(2)} - f'')(q)\|, \end{aligned} \quad (3.16)$$

which along with (3.15) gives the result of Theorem 3.1 for $r=1$. Since $(t^{(0)} - f)(u_i) = 0$

$$\begin{aligned} \|(t^{(0)} - f)(x)\| &\leq \max_i \left| \int_{u_i}^x (t^{(1)} - f')(q) dq \right| \\ &\leq \theta^* p \|(t^{(1)} - f')(q)\|. \end{aligned} \quad (3.17)$$

This completes the proof of Theorem 3.1.

4. Difference between two extrapolated splines

Considering two values u, v of h , we propose to compare in this section two extrapolated cubic splines in the classes $D(3, P, u)$ and $D(3, P, v)$ which are the interpolant of Theorem 2.1. In this section, we shall prove the following:

Theorem 4.1. Suppose s^h is 1 periodic spline of Theorem 2.1 interpolating to the periodic function f . Then, for $h=u, v > 0$,

$$\| (s^u - s^v)(x) \| \leq 2 |v - u| K(1/6, u, v) \| A^{-1}(u) \| \tag{4.1}$$

where $K(\theta, u, v)$ is a positive function which depends on θ, u and v .

Proof. For any function g , we define, the operation $\delta_{u, v}$ by $\delta_{u, v} g = g(u) - g(v)$ and for convenience, we write δ for $\delta_{u, v}$. Thus, we see that (3.6) impels

$$\theta \theta^* p^3 (s_i^u - s_i^v)(x) = R_i(x) \delta N_i - Q_i(x) A \delta N_{i-1}. \tag{4.2}$$

Rewrite the equation (3) for $h=u$ and $h=v$, respectively, we assume at the following equality.

$$A(u) \delta N = \delta F - N(v) \delta A. \tag{4.3}$$

Further, in view of (3.4), we have

$$\| A^v(u) \| \leq \{ 2u(p-u) p^n \}^{-1}. \tag{4.4}$$

Next we observe that the matrix δ_A has at the most three non-zero elements.

Thus, subtracting the matrix $A(u)$ from $A(v)$, we see that

$$\| \delta_A \| \leq 2 |v - u| p^5. \tag{4.5}$$

Also, we observe that

$$\| N(v) \| \leq \| A^{-1}(v) \| \| F(v) \|. \tag{4.6}$$

Further, we have

$$\| \delta F \| \leq 3/2 \| v - u \| (p^2 + 3p(v+u) + 2(v^2 + u^2 + 2uv) p^3) w(f, p). \tag{4.7}$$

Thus, combining (4.3) - (4.7), we have

$$\| \delta N \| \leq \| v - u \| K(1/6, u, v) \| A^{-1}(u) \|, \tag{4.8}$$

where $K(\theta, u, v)$ is a operator function which depends on θ , u and v .

Finally, in view of (4.2) and observing that

$$\max_i |R_i(x)| \leq p^3 / 36, \quad (4.9)$$

It completes the proof of theorem 4.1.

Conclusion

We have studied existence, uniqueness and convergence property of extrapolated cubic spline with multiple knots which interpolate a given function at two points $1/6$ and $5/6$ of a general choice of set of points interior to each mesh interval which includes some earlier results in this direction of particular choice.

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