BISHOP AND SILOV DECOMPOSITIONS: A STUDY ON VECTOR FUNCTION SPACES

Rahul Pratap Singh[1], Professor Chinta Mani Tiwari[2]
[1,2]Department of Mathematics, School of Science, MUIT, Lucknow

ABSTRACT

This paper has been devoted to the study of vector function spaces on X. We define the Bishop and Silov decompositions in several ways for vector function spaces. If A denotes a complex function space on X then the tensor product A $ B of A and Banach algebra B can be regarded as a vector function space on X. The concept of the slice product $A \widehat{\otimes} B$ of a function algebra A with a Banach algebra B has been defined earlier [1]. We extend the idea of A # B for a complex function space A on X. Then A # B also can be considered as a vector function space on X and $A \widehat{\otimes} B \subset A$ # B, Mainly, we concentrate our study to the decompositions and their properties for vector function spaces of the types $A \widehat{\otimes} B$ and A # B, where A denotes a complex function space on X. In particular, we show that the Bishop (Silov) decompositions for a complex function space A and the vector function space $A \widehat{\otimes} B$ are the same.
KEYWORDS: function spaces, function algenbra, decompositions, Bishop and Silov decompositions

INTRODUCTION

Let X be a compact Hausdorff space and A be a linear subspace of C(X) that contains the constant functions. A_R denotes the real functions in A, i. e, ,= A_R A ∩ $C_R(X)$.
The real [2] and complex [3] multipliers for A are defined respectively as,

M(A) = { f ∈ $C_R(X)$: f g ∈ A for every g ∈ A} and
N(A) = { f ∈ C(X) : f g ∈ A for every g ∈ A }.

DEFINITIONS [15]. (i) A subset K of X is said to be a set of antisymmetry in (FP)- sense or an (FP)-antisymmetric set for A if whenever f is in MCA|$_K$), then f is constant.

The collection of all maximal sets of antisymmetry in (FP)-sense forms a decomposition of X. We shall call this decomposition the Bishop decomposition in (FP)-sense for A and denote it by K_{FP} (A).
(ii) A set of constancy of M(A) is called an (FP) - Silov get for A.

The collection of all maximal (FP)-Silov sets forms a decomposition of X, called the Silov decomposition in (FP)-sense for A. We shall denote it by $F_{FP}(A)$.

DEFINITION 1: [3]. A subset K of X is said to be a set of antisymmetry in (E)-sense or an (E)-antisymmetric set for A if f ∈ N(A) and f |$_K$ is real-valued, then f is constant on K.

The collection of all maximal sets of antisymmetry in (E)-sense forms a decomposition of X. We shall call this decomposition the Bishop decomposition in CE)-sense for A and denote it by K_E (A).

In a similar fashion we can define the Silov decomposition with the help of N(A).

DEFINITION 2: A set of constancy of N $(A)_R$ is called an (E)-Silov set for A.

The collection of all maximal (E)-Silov sets forms a decomposition of X which we shall call the Silov decomposition in (E)-sense for A and denote it by $F_E(A)$.

Now onwards, we shall take A to be a function space on X, i.e., A is a closed subspace of C(X) with 1 ∈= A.

Remarks

(i)   M(A) and N(A) are closed subalgebras of A_R and A respectively, containing constants.
(ii)   M(A) = N(A))$_R$ and hence $F_{FP}(A) = F_E(A)$ which we shall denote by F(A). Since, for any subset K of X, (NCA|$_K$)$_R$ ⊂ MCA|$_K$), $k_{FP}(A) < k_E(A)$. Also, since N(A) is an algebra, k_E (A) < (A) < (In fact, $k_E(A)$ and k_E (A) are the usual Bishop and Silov decompositions for N(A)).

(iii)   Since N(A) is a closed subalgebra of $C_R(X)$ and k(A) is a decomposition consisting of sets of constancy of M(A), k(A) is an u.s.c. decomposition.

(iv)   If A is an algebra, then M(A) = A and for any subset K of X, $M(A)|_k)_R = (N|k)_R = (NA|k)_R$. Therefore, $k_E(A) = k_{FP}(A) = 9fCA)$, where 9CCA) is the usual Bishop decomposition for A. Also, F(A) will be the usual Silov decomposition for A.

(v)   If A is a subspace of C_R (X), then $k_E(A) = f_E(A)$.

(vi)   It has been shown by Feyel and Pradelle that $k_{FP}(A)$ satisfies the CD)-property for A [4, Theorem 1]. However, looking to the proof of the result, it is clear that the authors have proved that $k_{FP}(A)$ satisfies the stronger (GA)-property for A (that is, if $\mu \in b (A)^e$ then supp $\mu \subset K$ for some K e % CA) ; see Definition 0.1.10(iii)) . Hence it follows from Remarks O. 1. 1 1 (i), (ii) and (ii) above that $k_E (A)$ and k(A) also have the (GA),(S) and (D)-properties for A.

Recently Yamaguchi and Wada have defined the idea of a set of antisymmetry for a closed subspace A of C(X) which separates the points of X [5]. Their definition is exactly the same as that of [6]. They have also proved that the corresponding decomposition has the CGA)-property for A and its members are p-sets for a compare.

Paltineanu [2] has defined antialgebraic sets for a subspace of $C_R(X)$.

Let M be a closed subspace of $C_R(X)$ with 1 e M. For a subset S of X, define

G(S) = { f ∈ M: for each g e M, there exists h ∈ M such that fg = h on S }.

A subset S of X is called an antialgebraic set for M if f ∈ G(S) implies that f is constant on S.

The collection of all maximal antialgebraic sets for M is a decomposition of X.

Remark. Paltineanu [2] has shown that the above decomposition has the (GA)-property for M. However, it is easy to see that the Paltineanu's decomposition of X into maximal antialgebraic sets for a closed subspace of $C_R(X)$ coincides with the Bishop decomposition in (FP)-sense. We have already noted that the latter has the (GA)-property. Thus Paltineanu's result is a special case of the result of Feyel and Pradelle [6].

If there is no danger of confusion regarding the subspace A under discussion, then we shall write k_{FP} ,k_E and F instead of $k_{FP}(A)$, $k_E (A)$ and k(A) respectively.

Theorem 1. Let A be a function space on X and l be an u.s.c. decomposition of X with the (D)-property for A. Then k < l.

Proof. Let F ∈ k. For f ∈ C_R(X/ l), we have f o q ∈ C_R(X), where q: X→X/ l is the quotient map. Also, (f o q)$|g_S$ i constant, say α_S , for each S ∈ l. Hence f o q ∈ A, by the (D)-property. To show that f o q ∈ $(N(A)_R$, let h ∈ A. Then f o q} $h|_S = \alpha_S h|_S \in A|_S$ for all S ∈ l. Since l has the CD)-property for A, (f o q) h ∈ A. Hence f o q ∈ N(A) and so, as in the proof of Theorem 1, F ⊂ S for some S ∈ *l*.

The following corollary is immediate.

Corollary 1 (i) k_{FP}= F if and only if k_{FP} is u.s.c. and (ii) k_E = F if and only if if is u.s.c..

Note that if kFP is u.s.c., then k_{FP} - F and hence

$k_{FP} = k_E = F$. Thus if k_{FP} is u.s.c., then k_E is u.s.c..

Next corollary shows that the Bishop decompositions determine the Silov decomposition. The proof is similar to the proof of Corollary 5.1.13. Corollary 2.1.9. Let A and A2 be function spaces on X. Then

$k_{FP}(A_1)= k_{FP}(A_2) \rightarrow k(A_1)=F(A_2)$ and

$k_E(A_1)=k_E(A_2) \rightarrow k(A_1) = F(A_2)$.

The converse of the above corollary is not true.

We shall give an example in which $k_E(A_1) = k_E(A_2)$ but $k_{FP}(A_1) \neq k_{FP}(A_2)$

and also give example in which $k_{FP}(A_1) = k_{FP}(A_2)$ but $k_E(A_1) \neq k_E(A_1)$.

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Note that, by Remark and Theorem 1, if f is finite, then X - X. The following theorem shows that somewhat stronger result is true.

Theorem 2. If y is finite, then k_{FP}. = F

Proof. Suppose that F = { F_1,F_2 ,...,F_2 } Now, M(A) is an algebra and hence, as in the proof of Theorem 1.1.15, for each i = 1,2, ... ,n; there exists fi e MCA) such that f. = 1 on F. and f. = 0 on F_j. for every j ≠ i.

Fix F_j. let$_g$ ∈ M(A|F_j). Then there exists g ∈ A such that g|Fj = g. Since f_j ∈ M(A), g f_j. ∈ A. To show that g'fj. M(A), let h ∈ A. Then (g' f_j.)h| F_k = [(g' h) | F_k] =0 if k ≠ i and (g'fj)|hFi = (g'h)|Fi = g(h|F$_i$) ∈ A| F_i , as g e M(A| F$_i$). Since F has the (D) property for A, (g f) h e A. Hence g f_i. ∈ MCA). Therefore, g f$_i$ is constant on F$_i$. and hence g is constant. This proves that F is a set of antisymmetry in (FP)-sense for A. Consequently, F_{FP}=F

The proofs of the following two propositions are similar to the proofs of Propositions 1 and 2.
Proposition 1. Let S be a CR set for A. Then
$k_{FP}(A|_S) < k_{FP} \cap S,$ $k_E (A/) < k_E(A)$ and
$k/ (A_S) < F(A) \cap S.$
Proposition 2. If S is a CR set for A which is saturated with $k_{FP}(A)$ then $k_{FP}(A|_S) = k_{FP}(A) \cap S$
Remark 2.1.13. The Proposition 2.1.12 is not true for k_E (Take S = F).

Generalization of a result of Hayashi

We have shown that the Silov decomposition is the finest u.s.c. decomposition with the (D)-property for a function space on X. Hayashi [7] has proved that the Bishop decomposition is the finest Hausdorff decomposable decomposition} with the (S)-propert.y for a function algebra. We generalize the result for a function space on X. For the proof, we need certain properties of p-sets, which we discuss now.

Throughout this section, A denotes a function space on X. We begin with recalling some definitions.

A subset S of X is called a peak set for A if there exists f ∈ A such that f/$_S$ = 1 for all x ∈ X-S. The intersection of peak sets is called a generalized peak set for A. A closed subset S of X is called a p-set for A if A ∈ A$^\perp$, where μ_S (G) = μ(S ∩ G) for every Borel set G of X.

Remarks
(i)   Suppose S ⊂ T ⊂ is a p-set for A and S is a p-set for A|$_T$, then S is a p-set for A.
(ii)   Let i Sα : α ∈ ∧ } be a family of p-sets for A. Then S = ∩ |$_\alpha$ S |$_\alpha$ is a p-set for A [7, p.6].

Proposition 3. If S is a p-set for N(A), then S is a P-set for A.

Proof. Let μ ∈ T$^\perp$ and ε > 0. Then there exists an open set U such that S ⊂ U and | μ|(U-S) < ε Since S is a p-set for N(A), by Remark, S is an intersection of peak sets for N(A). So, there exists a peak set T for N(A) such that S ⊂ T ⊂ U [8]. Let f ∈ N(A) be a peaking function for T. Then f^n converges to X$_T$ boundedly and pointwise, where X$_T$ is the characteristic function of T. Let g ∈ A. Then gf^n ∈ A and hence ∫ T gdu| = ∫ X gd$_T$dμ = lim|n ∫ X gfμ = 0, But |∫ S gfμ -∫ Tgd μ. |< ε| | g | |This implies that ∫ S gdfμ = 0, ε as s is arbitrary. Since this is true for every g ∈ A, μ_s ∈ A^T. Having discussed the p-sets for A, we go to the main result of this section. We start with a lemma.

Lemma 1. If a decomposition ^ of X has the (D)-property for A, then ? has the (D)-property for N(A)

Proof. Let f ∈ C(X) and f|$_S$ ∈ (NA)| for all S ∈ . Also, let g ∈ A. Since ℓ has the (D)-property for A, it suffices to show that fg|s (A|s)$^-$ for all S∈ ℓ. Let S ∈ ℓ. Since f|s ∈ (N(A)s|)$^-$ there exists a sequence {h$_n$} in N(A) such that h$_n$ converges uniformly to f on S. Then gh$_n$ ∈ A and therefore, gh$_n$ ∈ A|s for each n. Since gh$_n$ converges to gf uniformly on S, fg|s ∈ (A|s|)$^-$.

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We now define a Hausdorff decomposable decomposition.

Definition [7, p.14]. Let Y be a topological space. Two points y_1 , $y_2 \in$ Y are said to be H-equivalent if $f(y_1) = f(y_2)$ for all $f \in C_R(Y)$, the set of all real-valued, bounded continuous functions on Y. The collection of all H-equivalence classes will be denoted by X(Y) ={Y_N}, and we define the decomposition $x_\alpha = \{ Y_\alpha \}$ of Y for each ordinal number a, by transfinite induction, as follows :

(i)   If $\alpha = 0$, then $-x_0 =\{ Y \}$;
(ii)   If a does not have the immediate predecessor, then $x_\alpha = \widehat{\beta < \alpha}\, x_\beta$, where $\widehat{\beta < \alpha}\, x_{\beta=}\{\cap \mid_{\beta < \alpha} Y_B :$: YB$\in$ XB;
(iii)   If a has the immediate predecessor β, then $x_a =\cup \mid_{Y_b \in x_\beta} x(Y_b)$.

The process of defining a new partition, as above, from the previous ones will eventually stabilize, since the number a is limited by the number of partitions of Y. Hence there exists a minimum ordinal number a such that $x_a=x_{\alpha+1}$. We denote this ordinal byσ(Y).The decomposition $x_{\sigma(Y)}$ is called the Hausdorff decomposition of Y. If $x_{\sigma(Y)}$ consists of singleton sets {y} for all y $\in$ Y, then Y is said to be

(a) Hausdorff decomposable space.

For a decomposition δ of X, let Y = X/ δ. For each ordinal a, define $\widetilde{X_\alpha} = \{ \widetilde{Y_\alpha}: Y_\alpha \in X_\alpha \} = \{ q^{-1}(Y_\alpha) : Y_\alpha \in X_\alpha \}$ where q : X —> X/ δ is the quotient mapping and X_α is the decomposition of Y as defined above. X_α is called the decomposition of X associated with the decomposition X_α of X/ δ. It is clear that $\tilde{x}_{\sigma(Y)}$ δ if and only if X/ δ is a Hausdorff decomposable space.

We say that a decomposition δ of X is Hausdorff decomposable if $x_{\sigma(x/\delta)} = \delta$.

It is clear that if X is a compact Hausdorff space and 3 is an u.s.c. decomposition of X, then δ is Hausdorff decomposable.

Finally, we recall that a decomposition δ of X has the (S)-property for A if for any p-set T which is saturated with δ, $\delta \cap$ T has the (D)-property for $(A/T)^-$.

Theorem 3. Let A be a function space on X and $\delta < k_{FP}$ be a decomposition of X with the (S)-property for A. Then , $\tilde{x}_{\sigma(y)} = k_{FP}$, where $\tilde{x}_{\sigma(y)}$ is the decomposition of associated with the decomposition $\tilde{x}_{\sigma(y)}$ of Y = X/ δ.

Proof. Let q be the quotient map from X onto Y and for each ordinal α, left $\tilde{x}_a := \{ \widetilde{Y}_a : Y_a \in x_a \}$ be the decomposition of X associated with the decomposition X_δ of Y. First we shall show that

(i) $\widetilde{Y}_a$ is a p-set for A, for each $Y_a \in$ x_a and it is saturated with δ:
(ii) $k_{FP} < \widetilde{x_\alpha}$.

IF a = 0, then $\widetilde{x_\alpha}$ = {X}- and hence clearly (i) and (ii) hold. We assume that (i) and (ii) hold for all $\beta <$ a. If a does not have an immediate predecessor, then by the definition, $x_a =$ $\bigwedge_{\beta < \alpha} x\beta$. Therefore, $\widetilde{x_\alpha} = \bigwedge_{\beta < \alpha} \widetilde{x_\beta}$, i.e

$\widetilde{Y}_a =\bigcap_{\beta \langle \alpha}\{ \widetilde{Y}_a \in \tilde{x}_\beta: \widetilde{Y}_b \subset \widetilde{Y}_a \}$. By Remark 5.2.1 (ii), $\widetilde{Y}_a$ is a p-set and it can be seen that $\widetilde{Y}_a$ is saturated with γ Thus (i) holds. Since $k_{FP} < \widetilde{x_\beta}$ for all $\beta < \alpha$ $k_{FP} < \bigwedge_{\beta < \alpha} x\beta=\widetilde{x_\alpha}$

Suppose a has the immediate predecessor β. Let $K \in X_{FP}$. Then $K \subset \tilde{Y}_a$ for some $\tilde{Y}_a \in$ k_{FP} .Then $K \subset \tilde{Y}_b$ for some $\tilde{Y}_b \in \tilde{x}_\beta$. Left $f \in C_R (Y_b)$ .Then f o q $\in C_R$ $\tilde{Y}_b$ and (f o q)| $\in C_R (\tilde{Y}_b)$ and (f o q)|$_E$ is constant for every $E \subset \tilde{Y}_b$, $E \in \delta$. Since δ has the (S)-property for A, $\cap \tilde{Y}_b$. has the <D)-property for A|$\widetilde{Y_b}$ Hence, by Lemma 2.2.3, $\delta \cap \tilde{Y}_b$. has the CD}-property for N(A|$\widetilde{y_b}$) Therefore,

f o q $\in$ N(A|$\widetilde{y_b}$) $\cap C_R(\tilde{Y}_b,)$ i.e. , f o q M(A|$\widetilde{y_b}$) which implies that (f o q)|$_K$ $\in$ M(A|$_K$). Hence (f o q)|$_K$ is constant. This holds for any f $\in C_R$ (Y$_b$). Thus q(k) $\subset$Y$_b$ for some $Y_\alpha \in X_\alpha$. So, we have K $\subset$ $\tilde{Y}_\alpha \subset$ $\tilde{Y}_\alpha$. and hence (ii) holds. Also, it can be seen that $\tilde{Y}_\alpha$ ($\tilde{Y}_\alpha \in X_\alpha$),is saturated with δ. Thus to prove (i) , by Remark **1** (i) , it suffices to show that $\tilde{Y}_\alpha$ is a p-set for A|$\tilde{Y}_\alpha$. In view of Proposition **2**, it is enough to prove

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that $\widetilde{Y}_\alpha$ is a p-set for N (A| $\widetilde{Y_b}$). Let Y_b/H be the quotient space of Y_b which is obtained by H-equivalence and p: $Y_b \to Y_b$. /H be the natural quotient mappping. Since Y_b/H is a compact Hausdorff space, singleton set $\{p(Y_a)\}$ in Y_b/H is an intersection of peak sets $\{S_a\}$ for $C_R Y_b$/H). But, if f $\in C_R(Y_b$/H), then f o p o q $\in C_R(\widetilde{Y}_b)$ and f o p o q)/$_E$ is constant for every E$\subset \widetilde{Y}_b$. , E $\in \delta$, Therefore, by the same argument as above, f o p o f o p o q $\in$ Thus $C_R(Y_b$/H) o p o q c N(A/$_{\widetilde{Y_b}}$) and so, { (p o q)$^{-1}$ (S$_a$)} are peak sets for N(A/$_{\widetilde{Y_b}}$). Since $\widetilde{Y}_b$ = (p o q)$^{-1}$ 4(p (p(Y$_a$)) is the intersection of { (p o q)$-$ (S$_a$)}, $\widetilde{Y}_\alpha$ is the intersection of peak sets or a p-set for N(A/$_{\widetilde{Y_b}}$) which proves (i).

Now, $X_{FP} <$ $\tilde{x}_{\sigma(Y)}$ If is strictly finer than $\tilde{x}_{\sigma(Y)}$ then there exists $\tilde{Y}_T \in \tilde{x}_{\sigma(Y)}$ such that $\widetilde{Y}_T = U_a$ K_a for certain $K_a \in k_{FP}$, i,e, $\tilde{Y}_T$ is not an (FP)-antisymmetric set for A. Hence there exists f $\in$ M (A|$_{Ka}$ such that f is not constant. But $f|_{Ka} \in (M (A|_{Ka})$ and so, $f|_{Ka}$ is constant for each $K_a \subset \tilde{Y}_T$. Therefore, f defines a nonconstant real function on $Y_T \in \in$ $\tilde{x}_{\sigma(Y)}$. This contradicts the definition of $\tilde{x}_{\sigma(Y)}$ and completes the proof.

Since X_{FP} has the (S)-property for A, by taking $\delta = k_{FP}$ in the above theorem, it follows that k_{FP} is Hausdorff decomposable.

Corollary 2. Let A be a function space on X and δ be a decomposition of X which has the (S)-property for A and $\delta < k_{FP}$. Then $\delta = k_{FP}$ if and only if δ is Hausdorff decomposable. Thus k_{FP} is the finest Hausdorff decomposable decomposition with the (S)-property for A.

Proof. As noted at the end of the last theorem, if $k_{FP}= \delta$, then δ is Hausdorff decomposable. Conversely, suppose that δ is Hausdorff decomposable. Then $\tilde{x}_{\sigma(Y)} = \delta$, where Y =X/ δ. But, by Theorem **2,** $\tilde{x}_{\sigma(Y)}= k_{FP}$ and hence $k_{FP} = \delta$.

If X is . a metrizable space, then the proof of Theorem 3 can be simplified with the help of the following proposition.

Proposition 4. Suppose that X is a metrizable space. Let δ be a decomposition of X with the (GA}-property for A. Then each E e 8 is a p-set for A.

Proof. We imitate the method of the proof of Theorem 3 in [9]. Let E $\in \delta$. It is enough to show that $\mu_E \in A^\perp$ whenever $\mu \in b(A^\perp)$. Let $\mu = b(A^\perp)^e$. Then there exists F $\in \delta$ such that supp μ $\subset$ F. We have E = F or E $\cap$ F = φ. Hence $\mu_E = \mu$ 0. Hence $\mu_E \subset A^\perp$ .let μ $\subset b(A^\perp)$. Since X is metrizableC(X) is separable and hence the weak* topology on M(X) is metrizable. Since b (A$^\perp$) is the weak* closed convex hull of b(A$^\perp$)^e in M(X), by Choquet's theorem [46, p.19], there exists a regular Borel measure X supported on $(A^\perp)^e$ such that f (μ)= $\int_x$ $fd\mu$ = $\int_{b(A^\perp)^e} f(\vartheta)d\lambda)(\vartheta)$ for f every f= $\in$ (xx). Now, E is a G_δ-set and hence $\int_E$ $fd\mu =\int_X$ $fx_e d\mu = \int_{b(A^\perp)^e}$ $(fx_e)(\vartheta)d\lambda(\vartheta)$ for every f $\in$ (xx). But for $\vartheta \in$ b(A$^\perp$)^e , $\vartheta_E = \vartheta$ or $\vartheta_E= 0$. Hence, for f $\in$ A, 0 = $\int_E$ $fd\mu$ = $\int_x$ $fd\mu$ = $\int_x$ $(fx_{E)}$ $d\vartheta = (fx_{E)}9)(\vartheta)$. It follows that $\int_E$ $fd\mu$ = 0 for each f $\in$ A, i.e., $u_E \in$ $A^{\perp}$".85

To see how the above proposition simplifies the proof of Theorem **5**, we observe that to prove (ii) for $\tilde{x}_\beta$ we need (i) and (ii) for $\tilde{x}_\beta$ $\beta <$ a. Now, if we assume (ii) for $\tilde{x}_\beta$, by taking $\delta = \tilde{x}_\beta$, in the above proposition, (i) immediately follows for . $\tilde{x}_\beta$, Hence (ii) can be established for$\tilde{x}_\beta$, (as already proved in the proof of Theorem **5**). Having proved (ii) for $\tilde{x}_\beta$, (i) for $\tilde{x}_\alpha$, also follows immediately from Proposition **5** and no separate proof for this (as given in the latter part of the proof of Theorem 5) is necessary.

Remarks. (i) If $k_E = k_{FP}$, then, since k_E k(N(A)), the usual Bishop decomposition for N(A), k_{FP} has the (S)-property for N(A). Conversely, if k_{FP} has the (S)-property for N(A), then, by Corollary 5.2.6 applied to N(A), $k_{FP} = k_{FP}$ (N(A) = k_E.as $k_{FP}<k_E$ Hence k_{FP} k_E if and only if k_{FP} has the (S)-property for N(A). We shall give an example where $k_{FP} < k_E$ example**1** (b) which shows that does not satisfy the <S)-property for N(A), in general.

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(ii) Since f = $\tilde{x}_1$ and k_{FP} = $\tilde{x}_{\sigma(Y)}$ for Y =X/k_{FP}, it follows from the proof of Theorem **2** that the members of f and k_{FP}, are p-sets for A. Since k_E =k(N(A)), by Proposition **2**, members of k_E are p-sets for A also.

   We give examples of decompositions which are strictly finer than k_{FP}

Examples 2. (i) Let, X = [0,1] x [0,10] and m be the Lebesgue measure on X. Let A = {f ∈ C_R(X): $\iint f$(x,y)u(x)dm(x,y) = 0 for all u ∈ C_R[0,10] } Then A is a closed subspace of C_R (X) and k_{FP}(A) = {F_x: 0,≤×≤1}, where F_x = { (x,y): 0 ≤×≤1}15 Example 6(d)]. Let δ {F_x: 0,<×≤1}U {(0,y)}: 0 ≤ y ≤ 1}. Then δ is a decomposition of X and δ >≦ k_{FP}(A) It can be checked that δ is Hausdorff decomposable. Hence, by Theorem **5**, δ does not have the (S)-property for A.

(ii) Consider the function algebra B of Example **6** (iii). Then B is an antisymmetric algebra, i.e., k(B) = {X}. Let δ be the decomposition consisting of maximal weakly analytic sets for B. Then we have seen that δ ≦ k(B) and δ has the (GA)-property for B. Hence δ has the (S)-property for B. Thus, by Corollary **2**, δ is not Hausdorff decomposable. Also, by Corollary 2, we can say that there is no function space A on X for which k_{FP}(A)= δ.

   We can define the essential set of a function space on X as we have defined it for a function algebra. Thus if I denotes the largest closed ideal of C(X) contained in A, then E (A) = {x ∈ X**:** f(x) = 0 for all f ∈ I} is called the essential set of A.

   The existence of such I can be shown as it is shown for function algebras.

Remarks

   (i) Since I ⊂ N(A), E(A) = E(N(A)), where E(N(A)) denotes the essential set of N(A). Hence, by Proposition **2**, E(A) is a p-set for A.

(ii) If f ∈ C(X) and $f_{E(A)}$= then f ∈ NCA).

(iii) If F is a closed set such that f ∈ C(X) and $f|_F$= 0 => f ∈ A, then E(A) ⊂ F.

   The following proposition can be proved exactly as in the case of a function algebra [29**]**.

   Proposition **3**. Let A be a function space on X and E(A) denote the essential set of A. Let P_{FP} P_E and P denote respectively the union of all singleton sets of (A) and F(A). Then E(A) = $\overline{X - P_{FP}}$= $\overline{X - P_E}$= $\overline{X - p}$.

Decompositions far the space of affine functions

   Let K be a compact convex subset of a locally convex topological vector space and OK denote the set of extreme points of K. Also, let A(K) denote the space of continuous, real-valued affine functions on K. Ellis [11] has discussed the Bishop and Silov decompositions for A(K). Since a function in A(K) is determined by its value on OK, it is natural to concentrate on the space A(K)|$\overline{\partial K}$, where $\overline{\partial K}$ denotes the closure of OK. It is easy to see that A(K)|$\overline{\partial K}$ is a closed subspace of $C_R\overline{\partial K}$) which contains constants. As we have noted earlier, Paltineanu [2] has defined the decomposition of X into maximal antialgebraic sets for a subspace of C_R(X). Also, Feyel and Pradelle [6] have defined the Bishop and Silov decompositions for a cone of C_R(X). In this section, we compare the decompositions for A (K)|$\overline{\partial K}$ defined by these authors ([3], [6], [2]).

   There is a natural association between a function algebra A and A (Z) for a suitable compact convex set Z. We study the relation between the decompositions for A and for A(Z) which we shall use to construct some examples in the next section.

Definition 1. The centre of A(K) is the set of all f ∈ A(K) such that for each g ∈ A(K), there exists h ∈ A(K) with (fg)|$_{OK}$ = h|$_{OK.}$ It is denoted by C(A(K)).

   It is clear that C(A(K))|$\overline{\partial K}$ forms a uniformly closed subalgebra of C_R($\overline{\partial K}$) and contains constants.

   For E ⊂ OK,$\overline{coE}$ denotes the closed convex hull of E. For the definition of a split face of K and the theory related to A(K), we refer to Alfsen [10].

Ellis [11] has defined the Bishop and Silov decompositions for A(K) as follows.

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Definitions 2. (i) A subset E of ∂K is said to be a set of antisymmetry if whenever f $\in$ A(K) and $f|_{\overline{C\partial E}}$ $\in$ C(A($\overline{C\partial E}$)) then $f|_E$ is constant.

If $\{S_\alpha : \alpha \in \wedge \}$ denotes the collection of all maximal sets of antisymmetry, then $S_\alpha = \partial E_\alpha$, where E_α is some closed split face of K, for each a c A. The family $\{ E_\alpha : a \in \alpha \in \wedge \}$is called the Bishop decomposition for A(K) and is denoted by f(A(K)).

(ii) The sets of constancy of C(A(K))| $|_{\overline{\partial K}}$ are $\{ \partial F_a : a \in \wedge \}$, where f_a's are closed split faces of K. These faces f_α are called the faces of constancy for C(A(K)). The family $\{f_\alpha: a \in \wedge \}$ is called the Silov decomposition for A(K) and is denoted by f(A(K)).,

We shall use δ and f for δ(A(k)) and δ(A(k)), The members of δ and f are pairwise disjoint and K $= \overline{co}(\cup_a f_a) = \overline{co}(\cup_a E_a)$. In fact, $\cup_a \partial f_a = \partial K$ and $\cup_\alpha \partial E_\alpha$ ∂K.

Ellis [11] has shown that f $\cap$ $\overline{\partial k}$ is a decomposition of $\overline{\partial k}$ and it has the (D)-property for A(k)| $\overline{\partial k}$. He has also shown that, in general, $\delta \cap$ $\overline{\partial k}$ does not cover $\overline{\partial k}$ and it may not have the (D)-property. He has also discussed conditions under which $\delta \cap$ does have the (D)-property for A(K)|$\overline{\overline{\partial k}}$

The following definition is due to Ellis [11, p.571].

Definition 3. A subset E of $\overline{\partial k}$ is called a weak set of antisymmetry if whenever f $\in$ and f satisfies the condition that for each g $\in$ A$\overline{(coE)}$, there exists h $\in$ with $fg|_E$ = h$|_E$, then f is constant on E.

The maximal weak set of antisymmetry is of the form $W_\alpha = T_a \cap \overline{\partial k}$, where T_a is a closed split face of K. The family w(A (K) of all maximal weak sets of antisymmetry for A(K)$|_{\overline{\partial K}}$ forms a decomposition of ∂K. Let F(A(K))=$\{ T_a : T_a \cap \partial$K $\in w$(A(K)$)\}$

As usual, we write w and f for w (AK)) and f(A('A)).

Remarks. (i) δ < f and f < f.

(ii) If ∂K is closed, then δ = T,

(iii) w has the (D)-property for A(K) $|_{\overline{\partial K}}$.

Paltineanu [2] has shown that the decomposition of $\bar{\partial} K$into maximal antialgebraic sets for A(K) $|_{\overline{\partial K}}$ coincides with the decomposition w of $\bar{\partial} K$ into maximal weak sets of antisymmetry for A(K) ∂K defined above. So, by Remark, it follows that w coincides with the Bishop decomposition in (FP)-sense for A(K) $|_{\overline{\partial K}}$, i.e.,w(A(K) =k$_{FP}$(A(K) $|_{\overline{\partial K}}$.

The following proposition shows that f$\cap$ $|_{\partial K}$ is the Silov decomposition for ACEO|^ as considered in section 1.

Proposition 1. f $\cap$ =$\overline{\partial K}$= f(A(K) $|_{\overline{\partial K}}$).

Proof. It is enough to show that M(A(K) $|_{\overline{\partial K}}$) = (A(K) $|_{\overline{\partial K}}$ where M(A(K) $|_{\overline{\partial K}}$) = {f $\in$ C$_R$($|_{\overline{\partial K}}$): fh $\in$ A(K $|_{\overline{\partial K}}$). Let f $\in$ M(A(K) $|_{\overline{\partial K}}$). Since A(K) $|_{\overline{\partial K}}$) contains, M(A(K) $|_{\overline{\partial K}}$ $\subset$ A(K) $|_{\overline{\partial K}}$ and so, there exists f $\in$ A(K) Then g$|_{\overline{\partial K}}$ $\in$ A(K) $|_{\overline{\partial K}}$ and therefore, f(g$|_{\overline{\partial K}}$) $\in$ A(K) $|_{\overline{\partial K}}$, i.e , $f^{'} g^{'}|_{\overline{\partial K}}$ $\in$ A(K) $|_{\overline{\partial K}}$. Hence $f^{'} \in$ C(A(K)) and f =$f^{'}$ $g^{'}|_{\overline{\partial K}}$ $\in$ C(A(K$|_{\overline{\partial K}}$). Thus M(A(K) $|_{\overline{\partial K}}$) $\subset$ C(A(K$|_{\overline{\partial K}}$.

Conversely, let f $\in$C(A(K)) and f$|_{\overline{\partial K}}$ = $f^{'}$. Then, clearly, $f^{'} \in$ C$_R$$\overline{(\partial K)}$. Let $g^{'} \in$ A(K) $|_{\overline{\partial K}}$. Then there is g $\in$ A(K) such that g $|_{\overline{\partial K}}$ $g^{'}$. But, then fg$|_{\overline{\partial K}}$ $\in$ A(K) $|_{\overline{\partial K}}$, as f $\in$ C(ACK)). Therefore, fg$|_{\overline{\partial K}}$ $\in$ A (K) $|_{\overline{\partial K}}$, i.e $f^{'} g^{'} \in$ Hencef$^{'}$=f$|_{\overline{\partial K}}$ $\in$ and consequently, C(A(K) $|_{\overline{\partial K}}$ M(A(K) $|_{\overline{\partial K}}$.

Thus the decomposition of $\overline{\partial K}$ into maximal weak sets of antisymmetry and the restriction of the Silov decomposition for A(K) to $\overline{\partial K}$ coincide with the Bishop and Silov decompositions for A(K) $\overline{\partial K}$ according to the Definitions 1 and 3.

It is natural to ask when the Bishop and Silov decompositions for ACK) coincide. Ellis [12] has

Corresponding author: **Rahul Pratap Singh and Professor Chinta Mani Tiwari**

answered this question in terms of weak centrality.

If F is a closed split face of K, then $F_\perp = \{\in A\ (K): f|_F = 0$ is called a near-lattice ideal.

(ii) For a compact convex set K, A(K) is said to be weakly central if whenever I and J are maximal near-lattice ideals in A(K) such that I ∩ C = J ∩ C, C being the centre of A(K), then I = J.

Theorem. If A(K) is weakly central,then the Bishop and Silov decompositions for A(K)) are equal.

In view of the above theorem, it is of interest to characterize the weak centrality of A(K). We give one such characterization in the following proposition.

Proposition 2. A (K) is weakly central if and only if A(K) satisfies the following property :

(∗) If G and H are minimal closed split faces of K contained in the same face of constancy for C, then G = H.

Proof. Assume that A(K) has the (∗)-property. Let I and J be maximal near-lattice ideals in A(K) with I ∩ C. = J ∩ C,

i.e., $G_\perp \cap C = H_\perp \cap C$ for closed split faces G and H in K. Now, G and H are compact convex sets and so, it can be checked that $G \cap \partial K = \partial K$ and $H \cap \partial K = \partial K$. By Krein-Milman theorem, $\partial G \neq \emptyset \neq$. Thus $G \cap \partial K \neq \emptyset$ and $H \cap \partial K \neq \emptyset$. Since $K = \overline{co}(U_\alpha F_\alpha)$. $F_\alpha \in f, \partial K \subset U_\alpha F_\alpha$) , So, $G \cap F_\alpha \neq \emptyset$ and $H \cap F_\beta \neq$ (for some α and β. Now, G, H, F_α F_β and are closed split faces of K implies that $G \cap F_\alpha$ and $H \cap F_\beta$ are also closed split faces of K [1, Proposition]. Also, we have $G \cap F_\alpha \subset G$ and $H \cap F_\beta \subset H$. Since $G_\perp = I$ and $H_\perp = J$ are maximal near-lattice ideals in A(K), G and H are minimal closed split faces of K [10, p.145]. Thus $G \cap F_\alpha = G$ and $H \subset F_\alpha$ $H \subset F_\beta$ and $H \subset F_\beta$. Suppose $a \neq \beta$. Then there exists f ∈ C and constants λ and δ such that $f|_{F_\alpha} = \lambda$ and a $f|_{F_\beta} = \delta, \lambda \neq \delta$. Since $G \subset F_\alpha$, f- $\lambda \in I$ and also f-$\lambda \in C$, i.e., f- $\lambda \in$ I∩C= = J∩C. So, f- $\lambda \in J$ which is a contradiction. Therefore, $\alpha=\beta$ and G and H are contained in the same face of constancy for C. Hence, by the (∗)-property of A(K), G = H or equivalently I = J.

Conversely, suppose that A(K) is weakly central and G and H are minimal closed split faces of K contained in the same face of constancy F_α for C. Let $G_\perp = I$ and $H_\perp = J$. Then I and J are maximal near-lattice ideals in A(K). Let f ∈ I ∩ C. Then $f|_G$ is constant, say λ. Therefore, $f|_G = \lambda = f|_H$. But f ∈ I implies that $f|_G = 0$. So, $f|_H = 0$. Thus f ∈ $H_\perp$= J. Hence InC e J∩C ⊂ C. By the same argument, we can show that J ∩ C ⊂ I ∩ C. Therefore, J ∩ C = J ∩ C. Since A(K) is weakly central, I = J or G = H. Hence A(K) satisfies the (*)-property.

Given a function algebra, a space of affine functions can be associated with it, as the following construction shows.

Let A be a function algebra on a compact Hausdorff space X and S denote the state space of A, i.e., S = {∅ ∈ A* : ∅(1) =‖∅‖ = 1} . Let Z = $\overline{co}$(S∪(-Is)), the weak* closed convex hull of S ∪ (-iS). Then Z is a weak* compact convex subset of A* . Asimow [13] has shown that θ: A — A(Z), defined by (ef)(z) = Ref(z), for z ∈ Z, is a topological isomorphism of A onto A(Z).

The Bishop and Silov decompositions for A and A(Z) are related as follows.

Theorem. Let A be a function algebra on X and Z = $\overline{co}$(Su (-iS)). Then K∈ k(A) (respectively f(A)) if and only if K = K ∩ X for some k ∈ | δ(A(Z)) (respectively f(A(Z)))

Since the members of δ (A(Z)) (and f(A(Z))) are disjoint, the above correspondence is one-to-one also. Hence the following corollary is immediate.

Corollary K(A) = F(A) if and only if δ(A(Z)) = (A(Z)).

Also, it is shown in [11, Proposition 4] that δ(A(Z)) = t(A(Z))

Corresponding author: **Rahul Pratap Singh and Professor Chinta Mani Tiwari**

4* Tensor product

We know that the tensor product of two function algebras is a function algebra. Also, we have seen that if A and B are function algebras on X and Y respectively, then $K(A \widehat{\otimes} B) = K(A) \times K(B)$ and $f\{A\widehat{\otimes} B\} = f(A) \times f(B)$. The tensor product can also be defined for function spaces. So, the natural question is 'what about the corresponding Bishop and Silov decompositions ? We prove that $K_{fp}(A\widehat{\otimes} B) = K_{fp}(A) \times K_{fp}(B)$, $K_E(A\widehat{\otimes} B) = K_E(A) \times K_E(B)$ and $f(A\widehat{\otimes} B) = f(A) \times f(B)$, where A and B are function spaces on X and Y respectively. As usual, $A \widehat{\otimes} B$ denotes the uniform closure of the space of all finite linear combinations of functions { $f \widehat{\otimes} g : f \in A, g \widehat{\otimes} B$ }•, where $f \otimes g$ is a function on X x Y defined by $(f \otimes g)(x,y)=f(x)g(y)$. It is easy to see that $A \widehat{\otimes} B$ is a function space on X x Y. For $f \in A \& B$, $f_x \in B$ for all $x \in X$ and $f_y \in A$ for all $y \in Y$, where $f_x(y) = f(x,y)$ $(y \in Y)$ and $f_y(x) = f(x.y)$ $(x \in x)$.

Lemma. Let A and B be function spaces on X and Y. Then

(i) $N(A) \widehat{\otimes} N(B) \subset N(C \widehat{\otimes} B)$ and

(ii) (ii) if $f \in N(C \widehat{\otimes} B)$, then $f_x \in N(B)$ for all $x \in X$ and $f_y \in N(A)$ for all $y \in Y$

Proof. (i) Let $f \otimes g \in N(A) \otimes N(B)$, where $f \in N(A)$ and $g \in N(B)$.Then, clearly, $f \otimes g \in C(X \times Y)$. To show that $f \otimes g \in N(A \widehat{\otimes} B)$, let $h \in A\widehat{\otimes} B$. Then $h = \sum_{i=1}^{n} \emptyset_i \otimes \psi_i$ where $\emptyset_i \in A$ and $\psi_i \in$ for all i. Thus $(f \otimes g)h = (f \otimes g)h \sum_{i=1}^{n} \emptyset_i \otimes \psi_i = \sum_{i=1}^{n} \emptyset_i f_{\emptyset_i} \otimes g_{\psi_i}$; Since $f \in N(A)$ and $g \in N(B)$, $f_{\emptyset_i} \in A$ and $g_{\psi_i} \in B$ for all i. Therefore, $(f \otimes g)h \in A ® B$. Now, if $h \in A\widehat{\otimes} B$, then there is a sequence $\{h_n\}$ in $A \otimes B$ such that $h_n \to h$ uniformly n non X x Y. But, as we have shown above, $(f \otimes g) h_n \in A \otimes B$ for n all n and hence $(f \otimes g)h \in A\widehat{\otimes} B$. Thus $f \otimes g \in N(A \widehat{\otimes}B)$ and $N(A) \otimes N(B) \subset N(A \widehat{\otimes} B)$. Since $N(A \widehat{\otimes} B)$ is closed, $N(A) \widehat{\otimes} N(B) \subset N(A) \widehat{\otimes} N(B)$.

(ii) Let $f \in N(A \widehat{\otimes} B)$ and $g \in A$. Fix $y \in Y$. Then $(f_yg\_)(x)= (f_yg\_)g(x) = f(x,y)((g \otimes 1) (x,y) = (f(g \otimes 1))y (x)$ for all $x \in X$. Since $f \in N(A \widehat{\otimes} B)$, $f(g \otimes 1))y \in A$. Therefore, $f_yg \in A$ and consequently, $f_y \in N(A)$. Similarly, we can prove that $f_x \in N(B)$ for $x \in X$.

Next, we prove the result regarding the decompositions for the tensor product.

Theorem. Let A and B be function spaces on X and Y respectively. Then

(i) $f(A\widehat{\otimes} B) = f(A) \times f(B)$;

(ii) $k_E((A\widehat{\otimes} B)=k_E(A)\times k_E(B)$ ands

(iii) $k_{FP}(A\widehat{\otimes} B) = k_{FP}(A) \times k_{FP}(B)$

Proof. (i) By Lemma 1 (i), $CN(A) \widehat{\otimes} N(B))_R \subset (N(A\widehat{\otimes} B)_R$. Therefore, $f(A \widehat{\otimes} B) = f(N(A \widehat{\otimes}B)) < f(N(A) \widehat{\otimes} N(B))$, which is equal to $f(A) \times f(B)$, by Theorem 1.1.6. Hence $f(A\widehat{\otimes}B) < f(A) \times f(B)$.

Conversely, let $G \in f(A)$ and $H \in f(B)$. It is enough to show that G x H is a set of constancy of $(N(A \widehat{\otimes} B)\}_R$. let $f \in (N(A \widehat{\otimes} B))_R$. Then, by Lemma 2.4.1(ii), $f_X \in (N(B))_R$ and $f_y \in (N(A))_R$ for each $x \in X$ and $y \in Y$. So, f is constant on H for each $x \in X$ and f_y is constant on G for each $y \in Y$ which implies that f is constant on G x H. Hence G x H is a set of constancy of $(N(A \widehat{\otimes}B)_R$. Consequently, we get $f(A) \times f(A) < f(A\widehat{\otimes}B)$.

(ii) Since $N(A) \widehat{\otimes} N(B) \subset N(A \widehat{\otimes} B)$, it is easy to verify that $f(N(A \widehat{\otimes}B)) < f f(N(A) \widehat{\otimes} (N(B)) = f(N(A)) \times f(N(B))$, by Theorem 1.1.3(iii), where k indicates the usual Bishop decomposition. Thus $k_E.(A\widehat{\otimes}B) <k_E (A) \times k_E (B)$.

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Conversely, let $G \in k(A)$ and $H \in k_E(B)$. To show that G x H is an (E)-antisymmetric set for $A \widehat{\otimes} B$, let $f \in N(A \widehat{\otimes} B)$ and $f|_{GxH}$ be real-valued. Then, by the same argument as in (i), we get $f|_{GxH}$ is constant. Hence $k_E(A)$ x $k_E(B) < k_E(A\widehat{\otimes}B)$.

Let $G \in k_{FP}(A)$ and $H \in k_{FP}(B)$. Also, let $f \in M(A\widehat{\otimes} B \, B|_{GxH})$ But $M(A\widehat{\otimes} B \, B|_{GxH}) \subset M((A/G)^- \widehat{\otimes} (B/H)^- \widehat{\otimes} = N(A/G)^- \widehat{\otimes} (B/H))^-_R$. Therefore, by Lemma 2.4.1(ii), $f_y \in (N(A/G)^-_R = M(A/G)$ and $f_x \in (N(B/H)^- = M(B/H)$ for each $x \in G$ and $y \in H$. Hence f_x is constant on H and f_y is constant on G for each $x \in G$ and $y \in H$. Thus f is constant. Therefore, G x H is an (FP)-antisymmetric set for $A \widehat{\otimes} B$. Hence $k_{FP}(A) \, k_{FP}(B) \, k_{FP}(A\widehat{\otimes} B)$.

Let $T \in k_{FP}(A\widehat{\otimes}B)$. It suffices to show that $\pi_1(T)$ is an (FP)-anti symmetric set for A, where π_1: X x Y —▶ X is the projection map. Let $f \in M \{A|_{\pi_1(T)}\}$. Then $f \widehat{\otimes} 1 \in M((A\widehat{\otimes}B|\boldsymbol{\pi_{1(T)xY}}))$ and so, $f \widehat{\otimes} 1|_T \in M(A\widehat{\otimes}B|_T)$. Since $T \in k_{FP}(A \widehat{\otimes} B)$, $f \widehat{\otimes} 1|_T$ is constant and hence f is constant which shows that $\pi^1(T)$ is an (FP)-antisymmetric set for A.

While we are discussing the tensor product of two subspaces, it would be of interest to know the relation between $A(K_1) \widehat{\otimes} A(K_2)$ and $A(CK_1xK_2)$, where K_1 and K_2 are compact convex subsets of the same locally convex topological vector space and ACK_1 x $K_2)$ is the set of all continuous, real-valued affine functions on K_1 x K_2.

Let $BA(K_{1 \, x} K_2)$ denote the set of all biaffine functions on $K_{1 \, x} K_2$, i.e., the set of all real-valued functions on $K_{1 \, x} K_2$ which are affine in each variable separately. Equivalently,

$BA(K_{1 \, x} K_2) = \{ f \in C_R(K_{1 \, x} K_2): f_x \in A(K_2)$ and $f_y \in A(K_1)$ for $x \in K_1, y \in K_2 \}$.

Remarks 2.4.3. (i) It is clear from the definition that $A(K_1) \widehat{\otimes} A(K_1) \subset BA(K_2$ x K2>.
(ii) It is easy to check that $A(K_1$ x $K_2) \subset BA(K_1$ x $K_2)$,

However, $A(K_1,) \widehat{\otimes} A(K_2,)$ may not be contained in $A(K_1$ x $K_2)$.

Example 1. Let $K_1=K_2$ [0,10]. Define f : k_1 —▶R by f(x) = x and g : K2 —- R by g(y) = y. Then f $\in A(K_1)$ and $g \in A(K_2)$. So, $f \otimes g \in A(K_1) \widehat{\otimes} A(K_2)$. But it is easy to see that $f \otimes g \in A(K_1x K_2)$.

We do not know whether $A(K1$ x $K_2) \subset A(K_2) \widehat{\otimes} A(K2)$.

We are now almost ready to give examples of function spaces for which at least two of the three decompositions k_{FP}, k_E and f are not equal. We need one more lemma for that purpose.

Lemma. Let B be a closed subspace of $C_R(X)$ with $1 \in B$ and $A = B+iB = \{f + ig: f, g \in B \}$. Then f(A) = f(B), $k_E(A) = k_E(B)$ and $k_{FP}(B)$.
Proof. First we show that NCA)
= N(B) = iN(B) ……

Let $f \in N(B) \subset B$ and let $g \in A$, i.e. , $g = g_1 + ig_2$, where g_1 , $g_2 \in B$. Then fg_1 and fg_2 are in B and so, $fg = fg_1 + ifg_2$ e B + iB = A. Therefore, $f \in N(A)$ and $N(B) \subset N(A)$. Since N(A) is a complex subspace of C(X) containing constants, N(B)+ $iN(B) \subset N(A)$. Conversely, let $f \in N(A) \subset A$. Therefore, $f = f_1 + if2$ with $f_1, f_2 \in B$. To show that $f_1, f_2 \in N(B)$, let $g \in B$. Then $fg \in A$, as $B \subset A$ and $f \in N(A)$. Thus $f_1g + ifg_2 \in B + iB$. Since $B \subset C_R(X)$ and f_1,f_2 g are in B, f_1g and f_2g are in B. Hence $f_1, f_2 \in N(B)$. Thus (1) holds.

From (1) it is clear that $(N(A))_R = N(B) = (N(B))_R$. So, ^CA) = f(B). Also, for any closed subset G of X, we get $N(A)|_G = N(B)|_G + iNCB)|_G$ and hence $(N(A) |_G)_R = N(B) |_G = (N(B) |)_{GR}$. Thus $f_E(A) = k_E(B)$. Let G be a closed subset of X. Since A = B + iB, $A|_G = B|_G + iB|_G$. So, as we have proved above, $N(A|_G) = iN(A|_G) + iNCB|G)$. Therefore, $M(A|_G) = (N(A|_G))_R = N(B|_G) = H(B|_G)$ and hence $k_{FP}(A) = k_{FP}(B)$.

The promised examples of subspaces now follow. We use the results of the previous section and technique of tensor product to construct the examples.

CONCLUSIONS

Corresponding author: **Rahul Pratap Singh and Professor Chinta Mani Tiwari**

In this paper, we study two Bishop type decompositions, defined by Feyel and Pradelle and by Edwards for a subspace A of COO. In section 1, we discuss the relation between these two types of Bishop decompositions and also their relation with the Silov decomposition. If A is a subspace of $C_R(X)$, then one of these Bishop decompositions coincides with the decomposition defined earlier by Paltineanu. The main theorem of section 2 states that the Bishop decomposition in (FP)-sense is the finest Hausdorff decomposable decomposition with the (S)-property for A. This generalizes a result of Hayashi. In section 3, we discuss the space A(fQ of continuous, real-valued, affine functions on a compact convex set K. In the last section, we prove that the Bishop and Silov decompositions of the tensor product of function spaces are the product of the corresponding decompositions. Finally, we use the results of section 3 and tensor product technique to give examples where at least one decomposition is different from the others.

REFERENCES

1. R. D.Mehta, Tensor product of Banach algebras and algebras of functions, Ph.D.Thesis, Sardar Patel University, 1977.
2. G. Paltineanu, Antialgebraic sets with respect to asubspace of Cr(X) , Rev. Roumaine Math. Pure Appl. 21(1976), 1391-1397.
3. D. A. Edwards, A short proof of a theorem of Machado, Math. Proc. Camb. Phil. Soc. 99(1986), 111-114.
4. D. Feyel et A. De La Praaelle, Sur certaines extensions du theoreme d'approximation de Bernstein, Pacific J. Math. 115(1984), 81-89.
5. S. Yamaguchi and J. Wada, On peak sets for certain function spaces, Tokyo J. Math. 11(2) (1988), 415-425.
6. D. Feyel et A. De La Praaelle, Sur certaines extensions du theoreme d'approximation de Bernstein, Pacific J. Math. 115(1984), 81-89.
7. M. Hayashi, On the decompositions of function algebras, Hokkaido Math. J. 1(1974), 1-22.
8. G. M. Leibowitz, Lectures on Complex Function Algebras, Scott Foresman and Co., Illinois, 1970.
9. I. Glicksberg, Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math. Soc. 105 (1962), 415-435.
10. E. M. Alfsen, Compact convex sets and boundary integrals, Springer-verlag, Berlin, 1971.
11. A. J. Ellis, Central decompositions and the essential set for the space ACK), Proc. London Math. Soc. 26(1973), 564-576.
12. .---------, Central decompositions for compact convex sets, Compositio Math. 30(3)11975), 211-219.
13. L. Asimow, Decomposable compact convex sets and peak sets for function spaces, Proc. Amer. Math. Soc. 25(1)(1970), 75-79.

Corresponding author: **Rahul Pratap Singh and Professor Chinta Mani Tiwari**