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Analysis and Estimation of Distributions using Linear Combinations of Order Statistics

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Abstract
It is well known that the computation of higher order statistics, like skewness and kurtosis (which we call C-moments) is very dependent on sample size and is highly susceptible to the presence of outliers. To overcome these difficulties, Hosking (1990) has introduced related statistics called L-moments. L-moments are expectations of certain linear combinations of order statistics. They can be defined for any random variable whose mean exists and form the basis of a general theory which covers the summarization and description of theoretical probability distributions and hypothetical tests for probability distributions. The theory of L-moments parallels the theory of conventional moments as this list of applications might suggest. The main advantage of L-moments over conventional moments is that L-moments, being linear functions of data, suffer less from effects of sampling variability and the probability density functions that are estimated from L-moments are superior estimates to those obtained from conventional moments (maximum likelihood estimates). L-moments are more robust than conventional moments to outliers in the data and enable more inferences to be made from small samples about an underlying probability distributions. L-moment derived distributions for real data examples appear to be more consistent sample to sample than pdf's determined by conventional means.

Key words: Kurtosis; L-Statistics; Moments; Order Statistics; Skewness; Estimation.

Introduction

Probability density functions of distributions are of central importance in science and engineering and certainly to us. For example, Mauricio sacchi (1996) shows how we can build objective functions using Baye's theorem, Allan Woodbury (1989) is involved in Bayesian interpolation and Tad Ulrych (1998) is examining probability density functions (pdf's) associated with certain distributions. An excellent preprint by Gouveia, de Moraes and Scales (1998) who write about the use of higher moments for the maximum entropy construction of Bayesian a priori pdf's presents clearly both the sue of derived pdf's as well as difficulties encountered in computing these functions from realizations. These difficulties prompted Hosking (1990) to formalize a method for the estimation of statistics that are measures of higher moments, skewness and kurtosis for example but that are more robust with respect to sample size and the presence of outliers. These measures are called L-moments are based on expectations of linear combinations of order statistics. Indeed, Vogel (1995) observes that the introduction of the theory of L-moments by Hosking is probably the single most significant recent advance in relating to our understanding of extreme events**.**
Hence, it is a standard statistical practice to summarize a probability distribution by its moments or cumulants. It is also common practice that in fitting of parametric distribution to a data set, to estimate the parameters by equating the sample moments to those of the fitted distribution. It is sometimes difficult to assess exactly what information about the shape of distribution is obtained by its moments of third and higher order. The numerical values of sample moments particularly when the sample is small, can be different from those of the probability distribution from which

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sample was drawn; and the estimated parameters of distribution fitted by the method of moments are less accurate than those obtained by the other estimation procedures such as maximum likely hood method.
The alternative procedure described here is based on L-moments which are analogous to the conventional moments but can be estimated by linear combinations of order statistics that is L-statistics.
L-moments have the advantage over conventional moments of being able to characterize a large number of distributions and when estimated from a sample of being more robust to the presence of outliers in the data. Also when compared with the conventional moments, L-moments are less subject to bias in estimation and approximately asymptotic normal distribution when the sample size is large. Parameter estimates obtained from L-moments are sometimes more accurate in small samples than even the maximum likelihood estimates.
Many statistical techniques are based on the use of linear combinations of order statistics. However, there has not heretofore been developed a unified approach to the use of order statistics for the statistical analysis of uni-variate probability distributions.
In this paper we present the characterization of probability distributions, the summary of observed data samples, the fitting of probability distributions to data and the testing of hypothesis about distribution form.

Theoretical summary of L-moments:

We present, briefly, the derivation of L-moments, as well as some details of the inverse problem that we solve to determine pdf's from L-moments.

L-moments

We are all very familiar with the first two moments, the mean and the variance. The next two moments, skewness and kurtosis are also familiar, but less so. In applied geophysics, at least, these moments are seldom used. Although their application has been somewhat limited in the past, the scale dependent phenomena that we are constantly encountered with demands that we look at our data in ever finer detail. This is where these statistics are of great importance. As we will see, conventional estimation of skewness and kurtosis has serious drawbacks. To obviate some of these drawbacks, Hosking (1990) formalized a method for their estimation that is based on expectations of linear combinations of order statistics. This approach originated with the work of Gini (1912) and was described by Kaigh and Driscoll (1987). Hosking's paper, a beautiful paper indeed, has had an explosive effect in some fields. Hoskings's paper has led to several papers that attempt to explain to those of whom, like us, are less well versed with the fine art of pure statistics, what these linear combinations actually mean and what advantageous properties they posses. We are referring to two papers in particular. Royston (1992) and Wang (1996). We present here a distillation that is brief and, we hope, to the point.

Definition of L-moments:

Define $X_{j:n}$ to be the j^{th} smallest moment in a sample of size n. The L-moments of X are defined by

$$\lambda_r = r^{-1} \sum_{j=0}^{r-1} (-1)^j \ (r-1cj) E[X_{r-j:r}] \ ; \ r=1,2,3.....$$

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Where the expectation of an order statistic may be written as

$$E[X_{j:r}] = \frac{r!}{(j-1)!(r-j)!} \int x \{F(x)\}^{j-1}\{1-F(x)\}^{r-j}\, dF(x)$$

Substituting this expression in above definition, expanding the binomials in F(x) and summing the coefficients of each power of F(x), the first four L-moments are defined by

$$\lambda_1 = E[X_{1:1}]$$
$$\lambda_2 = ½\ E[X_{2:2} - X_{1:2}]$$
$$\lambda_3 = (1/3)\ E[X_{3:3} - 2X_{2:3} + X_{1:3}]$$
$$\lambda_4 = (1/4)\ E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]$$

In fact the L in L-moments represents exactly this linearity. Wang (1996) gives the interpretation of L-moments as " One value in a sample gives the magnitude of X. Two samples, through their difference gives the variability of X. Three samples give an indication of the asymmetry of p(x) and finally four samples, tell us something about the peak of tails of p(x). "
The use of L-moments to describe probability distributions is justified by the
Theorem 1: (i) The L-moments λ_r, r=1,2,3,...... of a real valued random variable X exist if and only if X has finite mean (ii) A distribution whose mean exists is characterized by its L-moments $\{\lambda_r$; r=1,2,3.......}

Let us define the L-moment ratios of X to be the quantities

$$\tau_r = \lambda_r/\lambda_2 \quad r= 3,4,......$$

It is also possible to define a function of L-moments which is analogous to the coefficient of variation (CV). Bounds on the numerical values of the L-moment ratios and L-CV are given by the following theorem, proved in Hosking (1989).

Theorem2: Let X be a non-degenerate random variable with finite mean. Then the L-moment ratios of X satisfy $|\tau_r| < 1$, $r\geq3$. If in addition $X\geq0$ almost surely, then τ, the L-CV of X, satisfies $0 < \tau < 1$. The L-moments and L-moment ratios are useful quantities for summarizing a distribution. The L-moments are in some ways analogous to the (conventional) central moments and the L-moment ratios are analogous to moment ratios. In particular λ_1, λ_2, τ τ_3 and τ_4 may be regarded as measures of location, scale, skewnesAs and kurtosis respectively.

To see this consider equations, the definition of the λ_r as expectations of linear combinations of order statistics. Clearly λ_1, the mean is a measure of location. To interpret λ_2 consider the typical configuration of a sample of size 2, if the two values tend to be close together, as in Fig. (a), then λ_2 will be smaller than if they are far apart, as in Fig. (b). Thus λ_2 can be thought of as measuring the scale or dispersions of the distribution. Samples of size 3 are relevant to λ_3, in Fig. (c) a positive central second difference, tend to arise from positively skew distribution and Fig.(d) is more typical of distributions with negative skewness.

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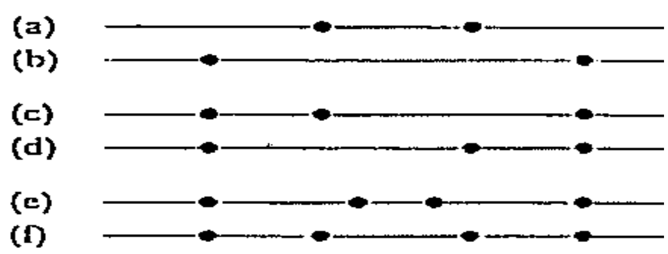
Symmetric distributions may have $\lambda_3 = 0$. Thus λ_3 may be thought of as measuring skewness, although not independently on scale. Similarly Fig. (e) and (f) explains samples of size 4. The Fig. (e), typical of a heavy tailed or sharply peaked distribution, has a large positive central third difference, while Fig (f) more typical of a flat or even U-shaped distribution, has a negative central third difference. Thus λ_4 is the third difference of expected order statistics of sample of size 4 and measures the same aspects of fourth central (conventional) moment. The L-moment ratios τ_3 and τ_4 are dimensionless analogues of λ_3 and λ_4 resepectively and are therefore plausible measures of skewness and kurtosis.

An alternative justification of these interpretations of L-moments may be based on the work of Oja (1981). Extending the work of Bickel and Lehmann (1975, 1976) and van Zwet (1964), Oja defined reasonable criteria for one probability distribution on the real line to be located further to the right (more dispersed, more skew and more kurtotic) than another. It follows immediately from Oja's work that λ_1 and λ_2 in Oja's notations $\mu_1(F)$ and ½ $\sigma_1(F)$ are measures of location and dispersion respectively. Hosking (1989) shows that τ_3 and τ_4 are by Oja's criteria, measures of skewness and kurtosis respectively.



Configurations of samples of sizes

In above we discussed that the main features of a probability distribution should be well summarized by the following four measures:

(1) The mean or L-location, λ_1
(2) The L-scale, λ_2
(3) The skewness, τ_3 and
(4) The kurtosis, τ_4

We now consider these measures, particularly τ_3 and τ_4 in more detail.

The L-moment measure of location is the mean, λ_1. This is well established and familiar quantity which needs no further description or justification here.

The L-scale, λ_2 is also long established in statistics. To compare λ_2 with the more familiar scale measure σ, the standard deviation, write

$$\lambda_2 = \tfrac{1}{2}\, E[X_{2:2} - X_{1:2}], \quad \sigma^2 = \tfrac{1}{2}\, E[X_{2:2} - X_{1:2}]^2$$

Both the quantities measure the difference between two randomly drawn elements of a distribution, but σ^2 gives relatively more weight to the largest differences.

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The L-skewness τ_3 is a dimensionless analogue of λ_3. By theorem 2, τ_3 takes the values between -1 and +1. These bounds are the best possible: they are approached aribitrarily closely as p ---> 0 or p --->1 in the Bernoulli random variable X_p with $P[X_p= 0] = p$, $P[X_p = 1] = 1-p$. Symmetric distribution have $\tau_3 = 0$.

However using Sillitto (1951), we write

$$\tau_3 = \{E[X_{3:3}] – 2E[X_{2:3}] + E[X_{1:3}]\} / \{E[X_{3:3}] – E[X_{1:3}]\}$$

shows that τ_3 is similar to the measure of skewness used by Bowley (1937)

$$B = (Q_3 -2Q_2 + Q_1)/ (Q_3 – Q_1)$$

Where Q_1, Q_2 and Q_3 are the quartiles of X. Skewness measures similar to B but based on quantiles other than quartiles have been used by Hinkley (1975) and Groeneveld (1984). In contrast, the conventional moment based measure of skewness,

$\gamma = E[X-E(x)]^3/ \{E[X-E(x)]^2\}^{3/2}$ is so sensitive to the extreme tails of the distribution that it is difficult to estimate accurately in practice when the distribution is markedly skew.

It is interesting to compare the skewness measures τ_3 and γ for various distributions: the comparison is made graphically in Fig.2. For symmetric distributions both τ_3 and γ are zero, and many near sysmmetric distributions have $\gamma = 6\ \tau_3$, but in general there is no relationship between γ and τ_3. Both γ and τ_3 may yield a large positive skewness either when a distribution has a heavy right tail or when a continuous distribution is reverse J shaped. The former case tends to yield particularly high values of γ relative to τ_3 since γ is more sensitive to the extreme tail weight of the distribution. Infact for some heavy tailed distributions γ approaches infinity while τ_3 has still a quite a modest value.

Kurtosis, as measured by the moment ratio

$$k = E[X – E(X)]^4 / \{E[X – E(X)]^2]^2$$

has no unique interpretation. It can be thought of as the 'peakedness' of a distribution, or as tail weight, but only for fairly closely defined families of symmetric unimodal distributions. L-Kurtosis, r_4 is equally difficult to interpret uniquely and is best thought of as a measure similar to k but giving less weight to the extreme tails of the distribution.

Given below table explains the first four L-moments and L-moment ratios for some common distributions. Values of r_3 and r_4 can be plotted to yield an L-moment ratio diagram.

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TABLE 1
L-moments of some common distributions†

Distribution	*F(x) or x(F)*	*L-moments*
Uniform	$x=\alpha+(\beta-\alpha)F$	$\lambda_1=\frac{1}{2}(\alpha+\beta),\ \lambda_2=\frac{1}{6}(\beta-\alpha),\ \tau_3=0,\ \tau_4=0$
Exponential	$x=\xi-\alpha\log(1-F)$	$\lambda_1=\xi+\alpha,\ \lambda_2=\frac{1}{2}\alpha,\ \tau_3=\frac{1}{3},\ \tau_4=\frac{1}{6}$
Gumbel	$x=\xi-\alpha\log(-\log F)$	$\lambda_1=\xi+\gamma\alpha,\ \lambda_2=\alpha\log 2,\ \tau_3=0.1699,\ \tau_4=0.1504$
Logistic	$x=\xi+\alpha\log\{F/(1-F)\}$	$\lambda_1=\xi,\ \lambda_2=\alpha,\ \tau_3=0,\ \tau_4=\frac{1}{6}$
Normal	$F=\Phi\left(\frac{x-\mu}{\sigma}\right)$	$\lambda_1=\mu,\ \lambda_2=\sqrt{\pi^{-1}}\,\sigma,\ \tau_3=0,\ \tau_4=30\pi^{-1}\tan^{-1}\sqrt{2}-9=0.1226$
Generalized Pareto	$x=\xi+\alpha\{1-(1-F)^k\}/k$	$\lambda_1=\xi+\alpha/(1+k),\ \lambda_2=\alpha/(1+k)(2+k),$ $\tau_3=(1-k)/(3+k),\ \tau_4=(1-k)(2-k)/(3+k)(4+k)$
Generalized extreme value	$x=\xi+\alpha\{1-(-\log F)^k\}/k$	$\lambda_1=\xi+\alpha\{1-\Gamma(1+k)\}/k,\ \lambda_2=\alpha(1-2^{-k})\Gamma(1+k)/k,$ $\tau_3=2(1-3^{-k})/(1-2^{-k})-3,$ $\tau_4=(1-6.2^{-k}+10.3^{-k}-5.4^{-k})/(1-2^{-k})$
Generalized logistic	$x=\xi+\alpha[1-\{(1-F)/F\}^k]/k$	$\lambda_1=\xi+\alpha\{1-\Gamma(1+k)\,\Gamma(1-k)\}/k,\ \lambda_2=\alpha\Gamma(1+k)\,\Gamma(1-k),$ $\tau_3=-k,\ \tau_4=(1+5k^2)/6$
Log-normal	$F=\Phi\left(\frac{\log(x-\xi)-\mu}{\sigma}\right)$	$\lambda_1=\xi+\exp(\mu+\sigma^2/2),\ \lambda_2=\exp(\mu+\sigma^2/2)\operatorname{erf}(\sigma/2),$ $\tau_3=6\pi^{-1/2}\int_0^{\sigma/2}\operatorname{erf}(x/\sqrt{3})\exp(-x^2)dx/\operatorname{erf}(\sigma/2)$
Gamma	$F=\beta^{-\alpha}\int_0^x t^{\alpha-1}\exp(-t/\beta)dt/\Gamma(\alpha)$	$\lambda_1=\alpha\beta,\ \lambda_2=\pi^{-1/2}\beta\Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha),\ \tau_3=6I_{1/3}(\alpha,2\alpha)-3$

†γ is Euler's constant; Φ is the standard normal distribution function; $I_x(p,q)$ is the incomplete beta function. Expressions for τ_4 for the gamma and log-normal distributions are given by Hosking (1986).

Estimation of L-moments:

L-moments are usually be estimated from a random sample drawn from an unknown distribution. Because l_r is a function of the expected order statistics of a sample size r, it is natural to estimate it by a U-statistics, i.e., the corresponding o function of the sample order statistics averaged over all subsamples of size r which can be constructed from the observed sample of size n. Let $X_1, X_2, \ldots X_n$ be the sample and $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ the ordered sample and define the r[th] sample L-moment to be

$$l_r=\binom{n}{r}^{-1}\sum_{1\le i_1<i_2<\ldots<i_r\le n}\cdots\sum r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}x_{i_{r-k}:n},\qquad r=1,2,\ldots,n$$

in particular

$$l_1=n^{-1}\sum_i x_i,$$

$$l_2=\frac{1}{2}\binom{n}{2}^{-1}\sum\sum_{i>j}(x_{i:n}-x_{j:n}),$$

$$l_3=\frac{1}{3}\binom{n}{3}^{-1}\sum\sum\sum_{i>j>k}(x_{i:n}-2x_{j:n}+x_{k:n}),$$

$$l_4=\frac{1}{4}\binom{n}{4}^{-1}\sum\sum\sum\sum_{i>j>k>l}(x_{i:n}-3x_{j:n}+3x_{k:n}-x_{l:n}).$$

U-statistics were introduced by Hoeffding (1948) and are widely sued in non-parametric statistics (see, for example, Fraser (1957) and Randles and Wolfe (1979)). Their properties of unbiasedness, asymptotic normality and some modest resistance to the influence of outliers make them particularly attractive for statistical inference.

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When calculating τ_r it is not necessary to iterate over all samples of size r, the statistic can be expressed explicitly as a linear combination of order statistics of a sample of size n, as in Blom (1980). Considering the U-statistic estimator of $E[X_{r:r}]$ and counting the occurrences in it of each $x_{i:n}$ shows the estimator may be written as rb_{r-1} where as in Hosking (1985),

$$b_r = n^{-1} \sum_{i=1}^{n} \frac{(i-1)(i-2)\ldots(i-r)}{(n-1)(n-2)\ldots(n-r)} x_{i:n};$$

Parameter estimation:

A common problem in statistics is the estimation, from a random sample of size n, of a probability distribution whose specification involves a finite number p of unknown parameters. Analogously to the usual method of moments, the 'method of L-moments' obtains parameter estimates by equating the first p sample L-moments to the corresponding populations quantities. Examples of parameter estimators derived using this method is given in table below.

TABLE 2

Parameter estimation via L-moments for some common distributions†

Distribution	*Estimators*
Exponential	(ξ known) $\hat{\alpha} = l_1$
Gumbel	$\hat{\alpha} = l_2/\log 2$, $\hat{\xi} = l_1 - \gamma\hat{\alpha}$
Logistic	$\hat{\alpha} = l_2$, $\hat{\xi} = l_1$
Normal	$\hat{\sigma} = \pi^{1/2} l_2$, $\hat{\mu} = l_1$
Generalized Pareto	(ξ known) $\hat{k} = l_1/l_2 - 2$, $\hat{\alpha} = (1+\hat{k})l_1$
Generalized extreme value	$z = 2/(3+t_3) - \log 2/\log 3$, $\hat{k} \approx 7.8590z + 2.9554z^2$, $\hat{\alpha} = l_2\hat{k}/(1-2^{-\hat{k}})\Gamma(1+\hat{k})$, $\hat{\xi} = l_1 + \hat{\alpha}\{\Gamma(1+\hat{k})-1\}/\hat{k}$
Generalized logistic	$\hat{k} = -t_3$, $\hat{\alpha} = l_2/\Gamma(1+\hat{k})\Gamma(1-\hat{k})$, $\hat{\xi} = l_1 + (l_2 - \hat{\alpha})/\hat{k}$
Log-normal	$z = \sqrt{(8/3)}\Phi^{-1}\left(\frac{1+t_3}{2}\right)$, $\hat{\sigma} \approx 0.999\,281z - 0.006\,118z^3 + 0.000\,127z^5$, $\hat{\mu} = \log\{l_2/\mathrm{erf}(\sigma/2)\} - \hat{\sigma}^2/2$, $\hat{\xi} = l_1 - \exp(\hat{\mu} + \hat{\sigma}^2/2)$
Gamma	(ξ known) $t = l_2/l_1$; if $0 < t < \frac{1}{2}$ then $z = \pi t^2$ and $\hat{\alpha} \approx (1 - 0.3080z)/(z - 0.058\,12z^2 + 0.017\,65z^3)$; if $\frac{1}{2} \leqslant t < 1$ then $z = 1 - t$ and $\hat{\alpha} \approx (0.7213z - 0.5947z^2)/(1 - 2.1817z + 1.2113z^2)$; $\hat{\beta} = l_1/\hat{\alpha}$

†γ is Euler's constant; Φ^{-1} is the inverse standard normal distribution function.

Exact distributions of parameter estimators obtained by the method of L-moments are in general difficult to derive. Asymptotic distributions can be found by treating the estimators as functions of sample L-moments and applying Taylor series methods, Serfling (1980), Hosking (1986) gives several examples of such results. For most standard distributions, this approach can be used to show that L-moment estimators of parameters and quantiles are asymptotically normally distributed and to find standard errors and confidence intervals. In applications we have found

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that asymptotic approximations are usually reliable for samples of size 50 or more: see for example Hosking (1985) and Hosking and Wallis (1987)

It is of interest to compare the method of L-moments with the asymptotically optimal methods of maximum likelihood. The method of L-moments is usually computationally more tractable than the method of maximum likelihood. The asymptotic standard errors of L-moment estimators, when compared with those of maximum likelihood estimators, usually show the method of L-moments to be reasonably efficient. For example, the efficiencies of the L-moment estimators of location and scale for the normal distribution are 100% and 97.8% respectively; for the Gumbel distribution the corresponding values re 99.6% and 75.6%. Asymptotic efficiencies of L-moment estimators tend to lower, but still reasonably high, for distributions with more than two parameters (Hosking, 1985).

Conclusion: The difference between L-moments over conventional moments is that L-moments, being linear functions of data, suffer less from effects of sampling variability and the probability density functions that are estimated from L-moments are superior estimates to those obtained from conventional moments (maximum likelihood estimates). No extension of L-moments to multivariate distributions is immediately apparent.

References:

1. Domenico, P.A. and F.W. Schwartz, 1997, Physical and Chemical Hydrogeology: Wiley Interscience.
2. Landwehr, J.M., Matalas, N.C., and Wallis, J.R., 1979, Probability-weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles: Water Resour. Res., 15, 1055-1064.
3. Gini, C, 1912, Vaariabilita e mutabilita, contributo alio studio delle distribuzioni e delle relazione statistiche: Studi Economico-Giurdici dell' Universita di Cagliari, 3, 1-158.
4. Gouveia, W., de Moraes, F.S. and Scales, J.A., 1998, Entropy, Inference and Information: submitted to J. Geophysical Int.
5. Hosking, J.R.M., 1990, L-moments: Analysis and estimation of distributions using linear combinations of order statistics: J. R. Stat. Soc, Ser. B, 52, 105-124.
6. Jaynes, E.T. 1957, Information theory and statistical mechanics: Physical Review, 106, 620-630.
7. Kaigh, W.D. and Driscoll, M.F., 1987, Numerical and data summary using O-statistics: American Statistician, **296,** 1702-1706.
8. Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P., 1992, Numerical Recipes in FORTRAN: the Art of Scientific Computing: Cambridge University Press.
9. Rietsch, E. 1977, The maximum entropy approach to inverse problem: Journal of Geophysics, 42, 489-506.
10. Rosa, A.L.R., and Ulrych, T.J., 1991, Processing via spectral modeling: Geophysics, 56, 1244-1251.
11. Shannon, C.E., 1948, A mathematical theory of communication: Bell Systems Technical Jour., **27,** 379-423.
12. Silverman, B.W., 1986, Density Estimation for Statistics and Data Analysis, Chapman & Hall, London.
13. Sudicky, E.A. 1986, A natural gradient experiment on solute transport in a sand aquifer: Spatial variability of hydraulic conductivity and its role in the dispersion process: Water Resources Research, **24(7),** 2069-2082.

Corresponding author: **J. Purushotham**
Submitted: 27Aug 2021, Revised: 09 Sep 2021, Accepted: 18 sep 2021, Published: Sep2021

14. Ulrych, T., 1998, The "whiteness" hypothesis: reflectivity, inversion, chaos and Enders, submitted to Geophysics.
15. Vogel, R.M., 1995, Recent advances and themes in hydrology: Rev. Geophys., **33,** Suppl.
16. Wang, Q.J., 1996, Direct sample estimators of L moments: Water Resour. Res., **32,** 3617-3619.
17. Woodbury, A. D., 1989, Bayesian updating revisited: Math. Geology, **21(3),** 285-308.

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