Classes Associated with Analytic and Harmonic Univalent Functions

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Abstract
Analytic and harmonic univalent functions are fundamental concepts in complex analysis and serve as key tools in various areas of mathematics, including complex function theory and geometric function theory. These classes of functions are particularly important in the study of conformal mappings and have widespread applications in physics, engineering, and other fields. In this abstract, we provide a brief overview of these classes and their significance.Analytic univalent functions are holomorphic (analytic) functions that map the unit disk onto a region in the complex plane in a one-to-one and onto manner. They are known for their role in preserving angles and shapes, making them essential in the study of conformal maps. These functions are often characterized by their coefficients in their Taylor series expansion and have important properties related to convexity, starlikeness, and univalency.harmonic univalent functions are solutions to the Laplace equation that are univalent in the unit disk. They arise in potential theory and have intriguing connections to analytic univalent functions through the so-called Grunsky inequalities. Harmonic univalent functions have applications in problems involving electrostatics and fluid dynamics.

Keywords:-Analytic Functions, Harmonic Functions, Univalent Functions, Complex Analysis

Introduction
The study of analytic and harmonic univalent functions constitutes a fundamental branch of complex analysis, offering valuable insights into the behavior of holomorphic and harmonic functions within the unit disk. These classes of functions play a pivotal role in the broader field of complex function theory, with profound applications in various domains, including geometry, physics, and engineering.Analytic univalent functions are characterized by their ability to map the unit disk conformally onto a region in the complex plane while maintaining a one-to-one and onto correspondence. Their importance lies in their capacity to preserve angles and shapes, making them indispensable in the analysis of conformal mappings. These functions are distinguished by properties such as convexity, starlikeness, and univalency, often revealed through their Taylor series coefficients. harmonic univalent functions emerge as solutions to the Laplace equation and

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are univalent within the unit disk. Their significance extends to potential theory, where they find applications in the modeling of electrostatic and fluid flow phenomena. Notably, harmonic univalent functions are connected to analytic univalent functions through the Grunsky inequalities, adding depth to the interplay between these two classes.

Importance of the Study

The study of analytic and harmonic univalent functions is of paramount importance in mathematics and its diverse applications. These classes of functions serve as the foundation for understanding complex mappings, geometry, and potential theory. Analytic univalent functions, in particular, play a pivotal role in the field of conformal mapping, preserving angles and shapes in geometric transformations. Their properties, such as convexity and starlikeness, find applications in optimization and various areas of mathematics. Harmonic univalent functions, on the other hand, contribute significantly to potential theory, enabling the modeling of electrostatics and fluid flow phenomena. Beyond mathematics, these functions have practical implications in fields like engineering, physics, and numerical analysis, making them indispensable tools for solving real-world problems. Moreover, ongoing research in these areas continues to expand our understanding of complex analysis and its broader applications.

Harmonic univalent functions

Harmonic univalent functions are a specialized class of complex-valued functions with two fundamental properties. First, they must be harmonic, meaning that their real and imaginary parts satisfy Laplace's equation, ensuring smooth behavior and the absence of singularities within a given domain in the complex plane. Second, these functions must be univalent, implying that they map points within their domain to unique points in their image, except possibly at the boundary. Harmonic univalent functions are essential in complex analysis, conformal mapping, and geometric function theory, playing a crucial role in various mathematical contexts, such as the study of Riemann surfaces and the theory of univalent functions. They offer a valuable tool for achieving one-to-one and conformal mappings between different domains, making them a powerful concept in the field of mathematics.

The study of the class H_S, which consists of complex-valued, harmonic, orientation-preserving, univalent functions in the open unit disc U, is a significant topic in complex analysis. These functions are normalized in such a way that $f(0) = f'(0) - 1 = 0$. Each function f in H_S can be

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represented as a sum of two components, h and g, both of which belong to the linear space H(U) of all analytic functions on U.

The pioneering work of Clunie and Sheil-Small, as documented in their paper [16], focused on the properties of H_S and some of its geometric subclasses. One of their notable findings was that while H_S itself is not a compact space, it exhibits normality with respect to the topology of uniform convergence on compact subsets of U. This is an important result in complex analysis as it helps understand the convergence behavior of functions in H_S.

Additionally, Clunie and Sheil-Small examined the subclass 0_SH of H_S, which comprises functions satisfying the property f(0) = 0. Remarkably, they established that this subclass is compact. This result has practical implications in the study of harmonic univalent functions and their behavior, especially in cases where the origin is a crucial point of interest.

The research conducted by Clunie and Sheil-Small on the class H_S and its subclasses has contributed significantly to our understanding of harmonic univalent functions and their properties, shedding light on the convergence and compactness properties of these functions within the unit disc U.

The Class HS(m,n,α)

Let Ur ={z: z < r,0< r ≤1} andU1 =U .A harmonic, complex–valued, orientation– preserving, univalent mapping f defined on U can be written as

$$f(z)=h(z)+\overline{g(z)},$$

Where,

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\ \ g(z)=\sum_{k=1}^{\infty}b_k z^k\ ,$$

are analytic in U.

Denote by HS(m, n,α) the class of all functions that satisfy the condition

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)(|a_k|+|b_k|)\le(1-\alpha)(1-|b_1|),$$

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where , 0 1and 0 | | 1. m∈ N n∈ N0 m > n ≤α<≤ b1 <1.

The class HS(m, n,α) with 0 b1 = will be denoted by HS° (m, n, α)

By setting specific values for the parameter, we can identify well-established subclasses that have been investigated by various researchers.

The classes HS HS(1,0,αα)≡(α) and HS HC (2,1,αα)≡HC(α) were studied by Öztürk and Yalcin

The classes HS(1,0,0)≡ and HS HC (2,1,0)≡ were studied by Avci and Zlotkiewicz

If h, g, H, G f (z)= h(z)+ g(z) and F(z)= H(z)+ G(z), then the convolution of f and F is defined to be the function.

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k},$$

while the integral convolution is defined by

$$(f \Diamond F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}.$$

The δ – neighbourhood of f is the set

$$N_\delta(f) = \left\{ F : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \le \delta \right\}.$$

In this case, let us define the generalized δ - neighborhood of f to be the set

$$N(f) = \left\{ F : \sum_{k=2}^{\infty} (k-\alpha)(|a_k - A_k| + |b_k - B_k|) + (1-\alpha)|b_1 - B_1| \le (1-\alpha)\delta \right\}.$$

Class of k-Uniformly convex functions

The class of k-uniformly convex functions is a subset of the broader class of convex functions in mathematical analysis. A real-valued function f(x) defined on a convex set X is considered k-uniformly convex if it satisfies the following definition:

For any two distinct points x and y in X, and for any λ in the interval [0, 1], the following inequality holds:

f(λx + (1 - λ)y) ≤ λf(x) + (1 - λ)f(y) - kλ(1 - λ) ||x - y||^2

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Here, ||x - y|| denotes the Euclidean distance between the points x and y, and k is a positive constant. This inequality essentially states that a k-uniformly convex function is "more convex" than a convex function, as it satisfies a stronger inequality.

K-uniformly convex functions have important applications in optimization theory, particularly in problems involving convex optimization and variational analysis. They provide a stricter notion of convexity, which can lead to more efficient algorithms and stronger convergence results in optimization problems. In practical terms, k-uniformly convex functions exhibit stronger curvature and smoother behavior compared to general convex functions, making them a useful concept in mathematical optimization and convex analysis.

Subclasses of harmonic univalent functions

Subclasses of harmonic univalent functions are specialized families of complex-valued functions that possess unique geometric and analytical characteristics. These subclasses are vital tools in complex analysis, conformal mapping, and geometric function theory. Notable among them are starlike and convex harmonic univalent functions, which preserve star-shaped and convex domains under mapping, respectively. Close-to-convex and quasiconformal harmonic univalent functions provide valuable insights into the preservation of convexity and quasiconformality properties. Additionally, subclasses with specific coefficient constraints and boundary behavior conditions offer insights into the analytical and geometric aspects of harmonic univalent functions. The study of these subclasses is instrumental in understanding how these functions transform domains, preserve geometric shapes, and meet various analytical criteria. Researchers employ these subclasses to solve real-world problems in physics, engineering, and mathematical modeling, making them indispensable in the broader context of complex analysis and the practical applications of harmonic univalent functions.

A function H f $\in$ S is said to be harmonic starlike of order α,($0 \leq \alpha < 1$), denoted by (α)S_H (α),

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) \geq \alpha\,, \qquad |z| = r < 1\,,$$

Or

$$\operatorname{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \alpha\,,$$

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and is said to be convex of order α (0 ≤α< 1) for z = r <1, if

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right)\right)\right)\geq\alpha,$$

Or

$$\operatorname{Re}\left\{1+\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\right\}\geq\alpha.$$

and denoted by HK (α) . These classes (α)∗ H S and HK (α) have been extensively studied by Jahangiri [46]. The case for α= 0 i.e. () * S_H (O)=$S_H$and HK(o) = HK respectively, the classes of starlike and convex functions in H S are given in [104] and for the case 1 α= 0= b_1 in

For (1<β≤ 4 3) and z ∈U

Let

$$M_H(\beta)=\left\{f\in S_H:\operatorname{Re}\frac{zh'(z)-\overline{zg'(z)}}{h(z)+\overline{g(z)}}<\beta\right\},$$

And

$$L_H(\beta)=\left\{f\in S_H:\operatorname{Re}\left(1+\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\right)<\beta\right\}.$$

Further, let V_H and U_H be the subclasses of S_H consisting of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k,$$

And

$$f(z)=z+\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k,$$

Respectively

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Research Problem

Analytic and harmonic univalent functions are two essential classes of complex-valued functions in the field of complex analysis, with wide-ranging applications in various mathematical and scientific disciplines. Research in this area seeks to understand the properties, behaviours, and interrelationships of these classes of functions. A promising research problem is to explore and establish connections between the classes associated with analytic and harmonic univalent functions.One possible avenue of investigation is to study the inclusion relationships between these classes. Researchers can explore conditions under which analytic univalent functions can be extended to harmonic univalent functions and vice versa. Understanding these inclusion relationships can provide insights into the interplay between harmonicas and analyticity, shedding light on the connections between two fundamental aspects of complex function theory.Another intriguing research direction is to examine the geometric properties of mappings associated with these classes. Investigating how these functions preserve angles, shapes, and conformal structures can lead to a deeper understanding of their behaviour and applicability in various domains, such as mathematical modeling, fluid dynamics, and conformal mapping.

Conclusion

The study of analytic and harmonic univalent functions represents a pivotal and multifaceted field in mathematics with far-reaching implications in various disciplines. Analytic univalent functions, by preserving angles and shapes in conformal mappings, are indispensable tools for solving problems in geometry, complex analysis, and optimization. Their properties, such as convexity and starlikeness, have profound geometric significance and influence diverse mathematical applications.Harmonic univalent functions, rooted in potential theory, provide crucial insights into physical phenomena like electrostatics and fluid dynamics. Their mathematical foundations are essential for modeling and solving complex problems in engineering and physics.Beyond their theoretical importance, these classes of functions hold practical value. They contribute to numerical analysis by aiding in the development of efficient algorithms and numerical simulations. They play a central role in understanding and solving real-world engineering and scientific challenges, making them indispensable in problem-solving across various domains.

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Future Research

Future research in the realm of classes associated with analytic and harmonic univalent functions promises to be a vibrant and evolving field with substantial implications for mathematics and its practical applications. As mathematicians continue to explore the interplay between analyticity and harmonic, the development of hybrid classes could open doors to novel functions with unique properties. The extension of these ideas to higher dimensions and multidimensional spaces is likely to contribute to the advancement of complex analysis, geometric function theory, and conformal mapping in multidimensional settings.the practical applications of these classes are boundless. Researchers can harness these mathematical tools to model and solve real-world problems in physics, engineering, image processing, and data analysis. The development of efficient computational methods and numerical techniques will further enhance our ability to study and utilize these classes effectively. Additionally, extremal problems within these classes offer intriguing challenges and the potential for discovering optimal mappings and transformations that can be applied to diverse fields.

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